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Cyclic Subgroups of the Simple Ternary Linear Fractional Group in a Galois Field.

By L. E. Dickson.

1. The present paper is an addition to the writer's article (*American Journal*, vol. XXII, pp. 231–252). It gives proofs of the results there stated in §§13–14 (pp. 249–251) and certain new theorems related to them. The question concerns the substitutions

$$x' = \alpha^r x, \quad y' = \alpha^s y, \quad z' = \alpha^{-r-s} z, \tag{1}$$

where α is a primitive root of the Galois Field of order p^n. Two cases arise according to the value of the greatest common divisor d of 3 and $p^n - 1$.

2. Suppose first that $d = 1$. The substitutions (1) are all powers of certain substitutions C of period $p^n - 1$ and having the form (1) in which the greatest common divisor $[r, s, p^n - 1]$ of $r, s, p^n - 1$ is unity. The cyclic group generated by C contains* $\phi(p^n - 1)$ substitutions of period $p^n - 1$. To determine which of them are conjugate under linear transformation, we must find when the powers C^a and C^b, where a and b are prime to $p^n - 1$, have the same sets of multipliers [the coefficients of x, y, z in (1)]. If a_1 be determined so that $aa_1 \equiv 1 \pmod{p^n - 1}$, whence $C^{aa_1} = C$, then C^a and C^b have the same multipliers if, and only if, C and C^{ba_1} do. It, therefore, suffices to determine when C and C^m have the same multipliers, where m is prime to $p^n - 1$ and $1 < m < p^n - 1$. But the sets of multipliers

$$\alpha^r, \alpha^s, \alpha^{-r-s}; \quad \alpha^{mr}, \alpha^{ms}, \alpha^{-mr-ms}$$

are identical apart from their order only in three cases:

(a). $\alpha^{mr} = \alpha^r, \alpha^{ms} = \alpha^s$. Then $r(m-1)$ and $s(m-1)$ are divisible by

* As usual, $\phi(m)$ denotes the number of integers $< m$ which are prime to m.

p^n-1. Since $[r, s, p^n-1]=1$, this requires that $m-1$ be divisible by p^n-1, contrary to the hypothesis $1<m<p^n-1$.

(b). $\alpha^{mr}=\alpha^s$, $\alpha^{ms}=\alpha^r$. We have the congruences modulo p^n-1:

$$mr\equiv s,\ ms\equiv r,\ m^2r\equiv r,\ m^2\equiv 1.$$

The fourth follows from the third, since $[r, s, p^n-1]=1$, so that r and s must each be prime to p^n-1. Inversely, if m be any solution of $m^2\equiv 1 \pmod{p^n-1}$, and if r be any integer less than and prime to p^n-1, and if s be determined by $s\equiv mr \pmod{p^n-1}$, then C and C^m have the same multipliers. Moreover, C is the r^{th} power of a substitution with the multipliers α, α^m, α^{-1-m}.

(c). $\alpha^{mr}=\alpha^s$, $\alpha^{ms}=\alpha^{-r-s}$, $\alpha^{-mr-ms}=\alpha^r$. Hence, if $M\equiv m^2+m+1$, both rM and sM, and consequently also M, are multiples of p^n-1. Since $m(m+1)$ is even, M is an odd number. Hence, p^n-1 must be odd, so that p^n is even and therefore $p=2$. The condition $M\equiv 0$ is equivalent to

$$(2m+1)^2\equiv -3, \qquad (\text{mod } p^n-1).$$

But* -3 is a quadratic residue of p^n-1 if, and only if, the latter be not divisible by 8, 9 or any prime of the form $6l+5$. Since $d=1$, $p^n-1=3^n-1$ or $3l+1$, the factor 3 cannot occur in p^n-1. Also n must be odd since $2^{2t}-1$ is divisible by 3. *Hence, solutions m occur if, and only if, $p^n=2^n$, n odd, and such that the prime factors of 2^n-1 are all of the form $6k+1$.*

If m be a solution of $x^2+x+1\equiv 0 \pmod{2^n-1}$, so is also $-1-m$. If $-1-m\equiv m$, then $m\equiv -\frac{1}{2}$, requiring $\frac{3}{4}\equiv 0 \pmod{2^n-1}$, whereas 3 is prime to 2^n-1, n odd. Hence, the solutions give rise, in sets of two, to the same substitution. The resulting substitution is the r^{th} power of a substitution having the multipliers α, α^m, α^{-1-m}. The latter substitutions belong to different cyclic groups; for, $(\alpha^m)^k=\alpha$ requires that $k=-1-m$.

It follows that the substitutions (1) all belong to the four types of cyclic groups of order p^n-1 given on p. 250 of the earlier paper.

As a first example, let $p^n=8$. There is one and but one cyclic group of each of the classes (i), (ii), (iii), and none of class (iv). The generators have respectively the sets of multipliers:

$$\alpha, \alpha^{-1}, 1;\quad \alpha, \alpha^2, \alpha^{-3};\quad \alpha, \alpha, \alpha^{-2}.$$

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* Gauss, "Disquisitiones Arithmeticae," Art. 120.

As a second example, let $p^n = 17$. Then $m^2 \equiv 1 \pmod{16}$ has the solutions $(m > 1) - 1, 7$ and 9, giving the sets of multipliers,

$$\alpha, \alpha^{-1}, 1; \quad \alpha, \alpha^7, \alpha^8; \quad \alpha, \alpha^9, \alpha^6.$$

The two cyclic groups of class (iv) are determined by the multipliers

$$\alpha, \alpha^2, \alpha^{13}; \quad \alpha, \alpha^3, \alpha^{12}.$$

3. Suppose next that $d = 3$, so that $p^n = 3t + 1$. Denote by Σ the homogeneous substitution with the multipliers $\alpha^t, \alpha^t, \alpha^t$. The quotient-group is obtained from the homogeneous group by making Σ correspond to the identity. Hence the set of exponents $r, s, -r-s$ in the substitution (1) may be replaced by the set $r+t, s+t, -r-s+t$ or by the set $r+2t, s+2t, -r-s+2t$, so that the exponents may be taken modulo t.

If t be prime to 3, *each substitution* (1) *is a power of some substitution of the form* (1) *and having the period* $p^n - 1$.

We may suppose that r and s are relatively prime; for if they have a common divisor g, the substitution (1) is the g^{th} power of a similar substitution. Hence one of the two, say r, is prime to $p^n - 1$. An integer r_1 may therefore be determined so that $rr_1 = 1 \pmod{p^n - 1}$, whence s/r is congruent to $sr_1 \equiv \rho$. Then $C = C_1^\tau$, where C_1 has the multipliers $\alpha, \alpha^\rho, \alpha^{-1-\rho}$ and is of period τ, the least integer such that $\tau(\rho - 1)$ and $\tau(\rho + 2)$ are divisible by $p^n - 1$. Hence $\tau = p^n - 1$ if, and only if, $\rho - 1$ is prime to 3. The theorem is therefore proved except for the case $\rho = 3l + 1$, when C_1 has the form

$$C_1: \quad x' = \alpha x, \quad y' = \alpha^{3l+1} y, \quad z' = \alpha^{-3l-2} z.$$

In the quotient-group, C_1 is of period $t = \frac{1}{3}(p^n - 1)$.

If, now, t be prime to 3, either t itself or else $2t$ will have the form $3m - 1$, so that $9m - 3$ is a multiple of $p^n - 1$. Either ΣC_1 or else $\Sigma^2 C_1$ has the form

$$x' = \alpha^{3m} x, \quad y' = \alpha^{3m+3l} y, \quad z' = \alpha^{3m-3l-3} z \equiv \alpha^{21m-3l-9} z,$$

which is the cube of the substitution of determinant unity,

$$x' = \alpha^m x, \quad y' = \alpha^{m+l} y, \quad z' = \alpha^{7m-l-3} z. \tag{2}$$

If l be not divisible by 3, the difference of the first two exponents in (2) is prime to 3 and therefore (2) is a power of a substitution of period $p^n - 1$. If l be divi-

sible by 3, C_1 is identical in the quotient-group with the substitution having the multipliers

$$\alpha^{3m}, \quad \alpha^{3m+3l} \equiv \alpha^{12m+3l-3}, \quad \alpha^{3m-3l-3} \equiv \alpha^{12m-3l-6}$$

and hence is the cube of the substitution of determinant unity

$$x' = \alpha^m x, \quad y' = \alpha^{4m+l-1} y, \quad z' = \alpha^{4m-l-2} z.$$

The difference of the first two exponents being prime to 3, this substitution, and therefore also C, is a power of one of period $p^n - 1$.

4. THEOREM.—*If t be divisible by 3, the substitution C_1 is contained in no cyclic group generated by a substitution* (1) *of period $p^n - 1$.*

Suppose that $C_1 = C^q$, C being a substitution of period $p^n - 1$ of the form (1). Since C_1 is of period t, the product tq must be divisible by $p^n - 1 \equiv 3t$. Hence q must be divisible by 3, so that $C_1 = S^3$, $S \equiv C^{q/3}$. Let S, when expressed in the form (1), have s as one of its exponents. Hence $3s \equiv 1 \pmod{t}$, whereas t is supposed to be divisible by 3.

5. The period of a substitution (1) is the least integer τ such that

$$\tau r \equiv \tau s \equiv \tau(-r-s), \pmod{p^n - 1}$$

Hence $\tau = p^n - 1$ if, and only if, $[r - s, 2r + s, p^n - 1] = 1$.* The condition may also be written $[r - s, 3r, p^n - 1] = 1$. For $d = 3$, $p^n - 1$ is divisible by 3. *Hence, for $d = 3$, a substitution* (1) *is of period $p^n - 1$ in the quotient-group if, and only if, $[r, s, p^n - 1] = 1$ and $r - s$ is prime to 3.*

We proceed to enumerate the number of substitutions P which are of the form (1) and have the period $p^n - 1$. In the notations of the earlier paper, p. 249, there are $\phi(p^n - 1)\,\psi(p^n - 1)$ sets of solutions r, s, each $< p^n - 1$, of $[r, s, p^n - 1] = 1$. We must exclude the E sets r, s for which $r - s$ is divisible by 3. Evidently E equals the number of sets of solutions r, $s \pmod{3t}$ of $[r, s, 3t] = 1$ for which $r = s + 3k$, where the integer k may be taken modulo t. Hence E is the number of sets of solutions $k \pmod{t}$ and $s \pmod{3t}$ of $[3k, s, 3t] = 1$, equivalent to the pair of conditions $[k, s, t] = 1$, $s \not\equiv 0 \pmod{3}$. Now $[k, \sigma, t] = 1$ has $\phi(t)\,\psi(t)$ sets of solutions $k, \sigma \pmod{t}$. Since s is to be determined modulo

* The greatest common divisor of a, b, c is designated $[a, b, c]$.

$3t$ so that $s \equiv \sigma \pmod{t}$ and $s \not\equiv 0 \pmod{3}$, s may equal any of the integers σ, $\sigma + t$, $\sigma + 2t$, not divisible by 3.

If t be prime to 3, one and only one of the integers σ, $\sigma + t$, $\sigma + 2t$ is divisible by 3, so that $E = 2\,\phi(t)\,\psi(t)$. Also

$$\phi(3t) = 2\,\phi(t),\ \psi(3t) = 4\,\psi(t).$$

Hence there are $6\,\phi(t)\,\psi(t)$ set of integers r, s, each $< 3t$, which lead to substitutions P. No two of the integers r, s, $-r-s$ are equal since their differences are all prime to 3. Allowing for their six permutations and for the equivalence of P, ΣP and $\Sigma^2 P$ in the quotient-group, *we obtain* $\frac{1}{3}\phi(t)\,\psi(t)$ *sets of multipliers giving non-conjugate substitutions of period* $p^n - 1$ *in the quotient-group.*

For example, if $p^n = 13$, there are four non-conjugate substitutions of period 12 in the quotient-group. Their sets of multipliers are

$$\alpha, \alpha^2, \alpha^9;\ \alpha, \alpha^3, \alpha^8;\ \alpha, \alpha^{-1}, 1;\ \alpha^2, \alpha^3, \alpha^7.$$

If t be divisible by 3, all or none of the integers σ, $\sigma + t$, $\sigma + 2t$ are divisible by 3, according as σ is or is not divisible by 3. Hence $E = 3E'$, where E' denotes the number of sets of solutions k, $\sigma \pmod{t}$ of $[k, \sigma, t] = 1$, $\sigma \not\equiv 0 \pmod{3}$. Let $3, q_1, q_2, \ldots$ denote the distinct prime factors of t. Of the $\frac{2}{3}t^2$ sets of two integers k, σ, each $< t$, with σ prime to 3, $\frac{2}{3}t^2/q_i^2$ sets have k and σ both multiples of q_i with σ prime to 3, $\frac{2}{3}t^2/q_i^2 q_j^2$ sets have k and σ both multiples of $q_i q_j$ with σ prime to 3, etc. Hence

$$E' = \tfrac{2}{3}\left\{t^2 - \sum_i \frac{t^2}{q_i^2} + \sum_{i,j} \frac{t^2}{q_i^2 q_j^2} - \sum_{i,j,k} \frac{t^2}{q_i^2 q_j^2 q_k^2} + \ldots\right\}.$$

Let $t = T3^\tau$, where T is prime to 3. Then the distinct prime factors of T are $q_1, q_2, \ldots$, so that

$$F(T) \equiv \phi(T)\,\psi(T) \equiv T^2 - \sum_i \frac{T^2}{q_i^2} + \sum_{i,j} \frac{T^2}{q_i^2 q_j^2} - \ldots.$$

Hence

$$E' = \tfrac{2}{3}\,t^2\,T^{-2}\,F(T) = 2.3^{2\tau-1}\,F(T),$$

$$\phi(p^n - 1)\,\psi(p^n - 1) = F(3t) = F(3^{\tau+1})\,F(T) = 8.3^{2\tau}\,F(T).$$

Excluding the $E = 3E'$ sets, there remain $6.3^{2\tau}F(T)$ sets of integers r, s, each $< 3t$, which lead to substitutions P.

If $t = 3^\tau T$, $\tau \geqq 1$, *there are* $M \equiv 3^{2\tau-1}F(T)$ *non-conjugate substitutions of period* $p^n - 1$ *in the quotient-group.*

For example, if $p^n = 19$, the $M = 9$ sets of multipliers leading to non-conjugate substitutions of period 18 are

$$\alpha, \alpha^{-1}, 1\ ; \quad \alpha^5, \alpha^{-5}, 1\ ; \quad \alpha^7, \alpha^{-7}, 1\ ; \quad \alpha, \alpha^2, \alpha^{-3}\ ; \quad \alpha^5, \alpha^{10}, \alpha^3\ ;$$
$$\alpha^7, \alpha^{14}, \alpha^{-3}\ ; \quad \alpha^{11}, \alpha^4, \alpha^3\ ; \quad \alpha^{13}, \alpha^8, \alpha^{-3}\ ; \quad \alpha^{-1}, \alpha^{-2}, \alpha^3.$$

The first three sets lead to substitutions belonging to the same cyclic group of order 18 ; likewise for the last six sets.

6. We next determine the cyclic groups of order $p^n - 1$ for the case $p^n - 1 = 3t$, t being prime to 3. Let C denote a substitution (1) in which $[r, s, 3t] = 1$ and $r - s$ is prime to 3. We seek the values of $m > 1$, m being less than and prime to $p^n - 1$, for which C and C^m are conjugate in the quotient-group. Since they must have the same multipliers, α^{mr}, α^{ms}, α^{-mr-ms} must be identical in some order with

$$\alpha^{r+ct}, \quad \alpha^{s+ct}, \quad \alpha^{-r-s+ct}, \qquad (c = 0, 1, \text{or } 2).$$

Since r and s do not have a factor in common with $p^n - 1$ and enter into the multipliers symmetrically, we may suppose that r is prime to $p^n - 1$. Three cases arise.

(a). If $\alpha^{mr} = \alpha^{r+ct}$, $\alpha^{ms} = \alpha^{s+ct}$, then $(r - s)(m - 1)$ is divisible by $p^n - 1$, so that $m - 1$ is divisible by 3. Since $r(m - 1)$ and $s(m - 1)$ are divisible by t, so is also $m - 1$. Hence, $m - 1$ must be divisible by $3t = p^n - 1$, whereas, $m - 1 < p^n - 1$. The case is, therefore, excluded.

(b). If $\alpha^{mr} = \alpha^{s+ct}$, $\alpha^{ms} = \alpha^{r+ct}$, then $(m + 1)(r - s)$ is divisible by $p^n - 1$, so that $m + 1$ is divisible by 3. Also,

$$m^2 r \equiv ms + mct \equiv r + (m + 1)ct \equiv r \pmod{3t}.$$

Hence $m^2 \equiv 1 \pmod{3t}$. For each solution of $x^2 \equiv 1 \pmod{t}$, there exists a single integer $m \pmod{3t}$ such that

$$m \equiv x \pmod{t}, \quad m + 1 \equiv 0 \pmod{3}.$$

Since 3 is a factor of $p^n - 1$ but not of t, there are $2^{\mu+\kappa-1}$ values of m to be considered, μ and $\varkappa$ being defined on p. 250 of the earlier paper.

Inversely, if r be any integer less than and prime to $3t$, and if $s \equiv mr - ct \pmod{3t}$, where c is either of the two residues modulo 3 which make $r - s \equiv ct - (m - 1)r$ prime to 3, then C has the same multipliers as $C^m \equiv C^m \Sigma^{cm}$.

Moreover, C is the r^{th} power of the substitution C' having the multipliers α, α^{m+kt}, $\alpha^{-1-m-kt}$, where k is determined from $rk \equiv -c \pmod 3$. It follows from the above determination of c that $m - 1 + kt$ is prime to 3. Hence, C' is of period $p^n - 1$. For $c \equiv 0$, $k \equiv 0 \pmod 3$, we reach the substitution C_1' with the multipliers α, α^m, α^{-1-m}. The second possible set of multipliers is

$$\alpha, \ \alpha^{m+kt}, \ \alpha^{-1-m-kt}, \quad (k \text{ and } -1-m-kt \text{ prime to } 3),$$

and defines a substitution C_2'. But if j be determined so that $(m-1)j \equiv -k \pmod 3$, we have

$$\begin{aligned} &m + jt \equiv m + kt + mjt, \quad m(m + jt) \equiv 1 + mjt, \\ &(-m-1)(m + jt) \equiv -1 - m - kt + mjt, \end{aligned} \qquad (\text{mod } 3t).$$

Hence $C_2' \equiv C_2' \Sigma^{mjt}$ is conjugate with $(C_1')^{m+jt}$. We may, therefore, confine the discussion to the $2^{\mu+\kappa-1}$ substitutions C_m with the multipliers α, α^m, α^{-1-m} where $m^2 \equiv 1 \pmod t$, $m + 1 \equiv 0 \pmod 3$. We next prove that they generate distinct cyclic groups. Indeed, $C_{m_1} = C_m^y$ requires that either α^{my} or else $\alpha^{(-1-m)y}$ shall be identical with α, α^{1+t} or α^{1+2t}. If this be true for α^{my}, then $my \equiv 1 \pmod t$ and, therefore, $y \equiv m \pmod t$, so that $C_{m_1} \equiv C_m^m$ is conjugate with C_m, whence $m_1 \equiv m \pmod t$. In the second alternative, $(-1-m)y \equiv 1 \pmod t$, so that $m + 1$ must be prime to t. Since $m^2 - 1$ is divisible by t, so must also $m - 1$ be divisible by t, whence $-2y \equiv 1 \pmod t$. Moreover, either α^y or else α^{my} must be identical with one of the quantities α^{m_1}, α^{m_1+t}, α^{m_1+2t}, and, therefore, y or else my must be a root of $x^2 \equiv 1 \pmod t$. But the latter has no root of the forms $-\frac{1}{2}$, $-\frac{m}{2}$, since 3 is prime to t. *Hence, there are* $2^{\mu+\kappa-1}$ *distinct cyclic groups of order* $p^n - 1$ *in each of which the substitutions of period* $p^n - 1$ *are conjugate in sets of two.* Their substitutions have in all $\frac{1}{2}\phi(p^n - 1)\, 2^{\mu+\kappa-1}$ distinct sets of multipliers.

(c). If $\alpha^{mr} = \alpha^{s+ct}$, $\alpha^{ms} = \alpha^{-r-s+ct}$, $\alpha^{-mr-ms} = \alpha^{r+ct}$, then

$$mr - s \equiv ms + r + s, \quad (m-1)(r-s) \equiv 3s \qquad (\text{mod } 3t)$$

requires that $m - 1$ be divisible by 3. Also,

$$r(m^2 + m + 1) \equiv (m + 2)ct \equiv 0 \qquad (\text{mod } 3t)$$

requires that $m^2 + m + 1 \equiv 0 \pmod{3t}$. As in §2, case (c), this congruence has solutions if, and only if, $p^n = 2^n$ and the prime factors of $t \equiv \frac{1}{3}(p^n - 1)$ are all of the form $6k + 1$. Also n must be even and prime to 3, since t is prime to 3.

Inversely, let the δ distinct prime factors of t be all of the form $6k+1$. Then $x^2+x+1\equiv 0 \pmod t$ has 2^δ solutions, each of which leads to an unique integer m such that $m\equiv x \pmod t$ and $m-1\equiv 0 \pmod 3$. Also, let r be any integer less than and prime to $3t$, and let $s\equiv mr-ct$, c being either of the two residues 1, 2 modulo 3 which make $r-s\equiv r(1-m)+ct$ prime to 3. Then the multipliers α^r, α^{mr-ct}, $\alpha^{-mr-r+ct}$ of C are identical with those of $C^m\equiv C^m\Sigma^{-ct}$, viz.:

$$\alpha^{mr-ct},\quad \alpha^{-mr-r-(m+1)ct},\quad \alpha^{r+(m-1)ct}.$$

Furthermore, C is the r^{th} power of a substitution C_1 with the multipliers

$$\alpha,\ \alpha^{m+kt},\ \alpha^{-m-1-kt}, \qquad [kr\equiv -c \pmod 3],$$

Since c is not divisible by 3, the same is true of $m-1+kt$, so that C_1 has the period p^n-1.

If m be a root of $x^2+x+1\equiv 0 \pmod{3t}$, a second root is $-m-1$. Hence C has the same multipliers as C^{-1-m}. Hence, in the cyclic group generated by C, the substitutions of period p^n-1 have the same sets of multipliers in groups of three.

Since $t=6j+1\equiv 1 \pmod 3$, $t^2\equiv t \pmod{3t}$ and the power $m+t$ of the first C_1 with the multipliers α, α^{m-t}, α^{-m-1+t} gives the second C_1 with the multipliers α, α^{m+t}, α^{-m-1-t}. Hence, there are $2^{\delta-1}$ sets of multipliers α, α^{m-t}, α^{-m-1+t} of substitutions C_1 conjugate with C_1^m, C_1^{-1-m}. A particular C_1 is conjugate only with its m^{th} or $(-1-m)^{\text{th}}$ powers, and is not conjugate with a different C_1. Indeed, $\alpha^{(m-t)y}=\alpha$, α^{1+t} or α^{1+2t} requires $my\equiv 1 \pmod t$, whence $y\equiv -1-m \pmod t$. *Hence, there are $2^{\delta-1}$ distinct cyclic groups of order p^n-1, in each of which the substitutions of period p^n-1 are conjugate in sets of three.* The number of distinct sets of multipliers involved is $\frac{1}{3}\phi(2^n-1)\,2^{\delta-1}$.

As a first example, consider the least possible value $n=14$ for which cyclic groups of the type considered in case (c) can occur. Then $t=\frac{1}{3}(2^{14}-1)=43\cdot 127$. The $2^{\delta-1}\equiv 2$ cyclic groups of that type are generated by substitutions with the multipliers

$$\alpha,\ \alpha^{1670},\ \alpha^{-1671};\quad \alpha,\ \alpha^{1885},\ \alpha^{-1886}.$$

As a second example, let $p^n=31$, so that $t\equiv 10$ is prime to 3. The quotient-group contains two special cyclic groups of order 30 falling under case (b); indeed, the congruences $m^2\equiv 1 \pmod{10}$, $m+1\equiv 0 \pmod 3$ give $m\equiv 11$ or $-1 \pmod{30}$. In place of the multipliers α, α^{11}, α^{18}, we may take α^{21}, $\alpha^{31}=\alpha$,

$\alpha^{38} = \alpha^8$. The multipliers of the substitutions of period 30 in the two special cyclic groups are respectively

$$\alpha, \alpha^8, \alpha^{21}; \quad \alpha^7, \alpha^{26}, \alpha^{27}; \quad \alpha^{13}, \alpha^{14}, \alpha^3; \quad \alpha^{19}, \alpha^2, \alpha^9;$$
$$\alpha, \alpha^{-1}, 1; \quad \alpha^7, \alpha^{-7}, 1; \quad \alpha^{13}, \alpha^{-13}, 1; \quad \alpha^{19}, \alpha^{-19}, 1.$$

There are two* general cyclic groups G_{30} generated by substitutions not conjugate with any of their powers. The exponents of the multipliers of their distinct substitutions of period 30 are given in the following two columns:

1, 2, 27	1, 5, 24
7, 14, 9	7, 5, 18
11, 22, 27	11, 25, 24
13, 26, 21	13, 5, 12
17, 4, 9	17, 25, 18
19, 8, 3	19, 5, 6
23, 16, 21	23, 25, 12
29, 28, 3	29, 25, 6

It was verified that every substitution (1) of period 30 is conjugate in the quotient-group with one of the substitutions whose multipliers are given above.

7. Suppose, lastly, that $p^n - 1 = 3t$, t being divisible by 3. Consider, first, the substitutions C of period $p^n - 1$. Proceeding as in §6, we find that case (c) is now excluded, since $m^2 + m + 1$ is never divisible by 9 and, therefore, not by $3t$. Two cases remain:

(a). As in §6, $m - 1$ must be divisible by t. Setting $m - 1 = kt$, we have $rkt \equiv ct \equiv skt \pmod{3t}$. Since $r - s$ is prime to 3, k must be divisible by 3, whereas $0 < m - 1 < p^n$.

(b). As in §6, we have $m + 1 \equiv 0 \pmod 3$, $m^2 \equiv 1 \pmod{3t}$. Of each pair of solutions $\pm m$ of the latter, one and only one makes $m + 1 \equiv 0 \pmod 3$. Hence there are $2^{\mu+\kappa-1}$ suitable values of m. Inversely, for any such m and any integer r less than and prime to $3t$, the values $s = mr - ct$ $(c = 0, 1, 2)$ make $r - s$ prime to 3, and hence lead to substitutions of period $p^n - 1$. Their multipliers are

$$\alpha, \alpha^m, \alpha^{-1-m}; \quad \alpha, \alpha^{m+t}, \alpha^{-1-m-t}; \quad \alpha, \alpha^{m-t}, \alpha^{-1-m+t}.$$

* In accord with result (c), p. 251, of the earlier paper.

If C_1 denote the substitution corresponding to the first set, the product $C_1^{m-t}\,\Sigma^m$ has the second set of multipliers and $C_1^{m+t}\,\Sigma^{-m}$ has the third set of multipliers.

Hence there are only $2^{\mu+\kappa-1}$ *cyclic groups of order* p^n-1; *their generators may be taken to have the multipliers* α, α^m, α^{-1-m}.

These cyclic group are all distinct. In fact, $\alpha^{my}=\alpha$, α^{1+t} or α^{1+2t} requires $y\equiv m \pmod t$; $\alpha^{(-1-m)y}=\alpha^{1+ct}$ requires $(m+1)\,y\equiv -1 \pmod t$, whereas $m+1$ and t have the common divisor 3. The results may be stated as at the bottom of p. 251 of the earlier paper.

8. It remains to determine the cyclic groups of order $t\equiv\frac{1}{3}(p^n-1)$, t being divisible by 3, which are not contained in any of the cyclic groups of order p^n-1. They may be generated by substitutions C of the form (1) in which $[r, s, 3t]=1$ and $r-s$ is divisible by 3. Inversely, every such substitution C is of period t. By §5, the number of sets r, s, each $<3t$, is $E\equiv 2\,3^{2\tau}F(T)$, where $t=3^\tau T$, T being prime to 3. To determine every substitution C which is conjugate with some of its powers C^m, $1<m<t$, we treat the three cases (a), (b), (c) of §6.

(a). This case is to be excluded since $r(m-1)$, $s(m-1)$, and, therefore, also $m-1$, must be divisible by t.

(b). Here m^2-1 is divisible by t and, therefore, by 3^τ. Also $r-s\equiv r(1-m)+ct$ is divisible by 3, so that $m-1$ is divisible by 3. It follows that $m+1$ is prime to 3 and therefore that $m-1$ is divisible by 3^τ. Inversely, if x be any solution of $x^2\equiv 1 \pmod T$ and m be determined by the conditions

$$m\equiv x \pmod T,\quad m\equiv 1 \pmod{3^\tau},$$

then $m^2\equiv 1 \pmod t$ and $m-1$ is divisible by 3. Hence there are $2^{\mu+\kappa-1}-1$ such integers m, $1<m<t$. For each m and r the condition

$$r(m^2-1)/t\equiv(m+1)\,c \pmod 3$$

determines c modulo 3. If m^2-1 is divisible by $3t$, then $c\equiv 0$; if $m^2-1=(3j\pm 1)\,t$, then $c\equiv\mp r$. In the former case, C has the multipliers

$$\alpha^r,\ \alpha^{mr},\ \alpha^{-r-rm}\qquad [m^2\equiv 1 \pmod{3t}],$$

and is conjugate with C^m. In the latter case, C has the multipliers

$$\alpha^r,\ \alpha^{mr\pm rt},\ \alpha^{-r-mr\mp rt}\qquad [m^2\equiv 1\pm t \pmod{3t}],$$

and is conjugate with C^m in the quotient-group. In either case, C is the r^{th} power of a substitution of period p^n-1 having the distinct multipliers α, α^{m+kt},

$\alpha^{-1-m-kt}$ ($k = 0, 1$ or 2). *Hence there are* $2^{\mu+\kappa-1} - 1$ *of these special cyclic groups of order* t. In all, they contain

$$\tfrac{1}{2}\phi(t)(2^{\mu+\kappa-1}-1) \equiv 3^{\tau-1}\phi(T)(2^{\mu+\kappa-1}-1) \qquad (3)$$

sets of unequal multipliers of substitutions of period t.

(c) Since $r - s \equiv r(1-m) \equiv 0 \pmod 3$, $m - 1$ is divisible by 3. Hence, $r(m^2+m+1)$ and, therefore, m^2+m+1 must be divisible by $3t$, whereas, it is not divisible by 9. The case must be excluded.

In $3\phi(3t)$ of the E sets $r, s, -r-s$, two of the three integers are equal. The remaining sets give rise to

$$\tfrac{1}{18}[E - 3\phi(3t)] \equiv 3^{2\tau-2}\phi(T)\psi(T) - 3^{\tau-1}\phi(T)$$

sets of unequal multipliers of substitutions which are distinct in the quotient-group. Excluding the sets (3), and dividing the resulting number by $\phi(t)$, we obtain

$$\tfrac{1}{2}[3^{\tau-1}\psi(T) - 2^{\mu+\kappa-1}],$$

as the number of cyclic groups of order t whose substitutions of period t are never conjugate with any of their powers. In addition to these, there are $2^{\mu+\kappa-1} - 1$ special cyclic groups determined above and one cyclic group of order t generated by the substitution with the multipliers $\alpha, \alpha, \alpha^{-2}$.

Example I.—For $p^n = 19$, there are two cyclic G_6 not in the cyclic G_{18} generated by substitutions (1). Their substitutions of period 6 have the multipliers

$$\alpha, \alpha, \alpha^{-2}; \quad \alpha^5, \alpha^5, \alpha^{-10}; \quad \alpha, \alpha^4, \alpha^{13}; \quad \alpha^5, \alpha^2, \alpha^{11}.$$

Example II.—For $p^n = 37$, there are four cyclic G_{12} not in the cyclic G_{36}. The multipliers of their substitutions of period 12 are

$$\begin{array}{llll}
\alpha, \alpha, \alpha^{-2}; & \alpha^5, \alpha^5, \alpha^{-10}; & \alpha^7, \alpha^7, \alpha^{-14}; & \alpha^{11}, \alpha^{11}, \alpha^{-22}; \\
\alpha, \alpha^4, \alpha^{-5}; & \alpha^5, \alpha^{-16}, \alpha^{11}; & \alpha^7, \alpha^{-8}, \alpha; & \alpha^{11}, \alpha^8, \alpha^{17}; \\
\alpha, \alpha^{10}, \alpha^{-11}; & \alpha^5, \alpha^{14}, \alpha^{17}; & \alpha^7, \alpha^{-2}, \alpha^{-5}; & \alpha^{11}, \alpha^2, \alpha^{-13}; \\
\alpha, \alpha^{16}, \alpha^{-17}; & \alpha^5, \alpha^8, \alpha^{-13}. &&
\end{array}$$

The last two substitutions are conjugate with their seventh powers.

9. We give a summary of the results of §§3–8, adding the number of conjugate groups of each type. For $p^n = 3t+1$, the order N of the simple linear fractional group G is

$$N = \tfrac{1}{3}(p^{3n}-1)(p^{2n}-1)p^{3n}.$$

The substitutions (1) generate cyclic groups which are subgroups of the following system of distinct cyclic groups:

For $t \equiv \frac{1}{3}(p^n - 1)$ prime to 3:

$2^{\mu+\kappa-1}$ sets each of $\dfrac{N}{{}^2(p^n-1)^2}$ conjugate G_{p^n-1},

$2^{\delta-1}$ sets each of $N/({}^1p^n - 1)^2$ conjugate G_{p^n-1},

$\frac{1}{6}\psi(t) - \frac{1}{3}2^{\delta-1} - \frac{1}{2}2^{\mu+\kappa-1}$ sets each of $N \div \frac{1}{3}(p^n-1)^2$ conjugate G_{p^n-1},

the second set occurring if, and only if, $p^n = 2^n$, n being even and $\frac{1}{3}(2^n - 1)$ having only prime factors (δ in number) of the form $6k + 1$.

For t divisible by 3:

$2^{\mu+\kappa-1}$ sets each of $N \div \frac{2}{3}(p^n - 1)^2$ conjugate G_{p^n-1},

$\frac{1}{2}3^{\tau-1}\psi(T) - \frac{1}{2}2^{\mu+\kappa-1}$ sets each of $N \div \frac{1}{3}(p^n - 1)^2$ conjugate G_{p^n-1},

$2^{\mu+\kappa-1} - 1$ sets each of $N \div \frac{2}{3}(p^n - 1)^2$ conjugate G_t,

$\frac{1}{2}3^{\tau-1}\psi(T) - \frac{1}{2}2^{\mu+\kappa-1}$ sets each of $N \div \frac{1}{3}(p^n - 1)^2$ conjugate G_t,

one set of $N \div \frac{1}{3}(p^{2n} - 1)(p^{2n} - p^n)$ conjugate G_t.

THE UNIVERSITY OF CHICAGO, *March*, 1901.

*Curves of Triple Curvature.**

JAMES G. HARDY.

The present paper is written to bring together and, in some measure, to add to the results which have been obtained concerning curves L of triple curvature. Equations of motion for systems in a four-dimensional space have been deduced and used to introduce the notion of an instantaneous plane of rotation. The derivation of the equations of motion is not new but it seemed best for the sake of clearness to retain it. By constructing the principal tetrahedroid at a point of a curve of triple curvature and studying its motion by means of the kinematical equations obtained, geometrical interpretations of the six rotations and also a set of formulæ corresponding to the Serret-Frenet formulæ for curves of double curvature have been arrived at. These formulæ have been applied to the study of curves L and, in particular, of the osculating hypersphere and the locus of its centers.

NOTE. Several of the results of this paper were contained in a paper read by me in 1898 before the Mathematical Seminary of the Johns Hopkins University. The articles by Professor Lovett and Mr. Hatzidakis in the American Journal of Mathematics, vol. XXII, have appeared since then.

THE MOTION OF SYSTEMS IN A SPACE OF FOUR DIMENSIONS.

Velocities in an invariable system.

1. To discuss the motion of a system in a space of four dimensions we proceed as in the discussion of a similar problem in three dimensional space. The motion will be referred to a set of four rectangular axes OX, OY, OZ, OW

* Conf. Brunel, Mathematische Annalen, XIX, p. 48; Pirondini, Giornali di Matematiche, XXVIII, p. 237; Piccioli, Giornali di Matematiche, XXXVI, p. 273.

through a fixed point O. Imagine a second set of four rectangular axes, having the same orientation as the first, invariably connected with the moving system, and whose origin o is one of its points. To determine the position of the system at any instant it is sufficient to know, in terms of the time, the coordinates X_0, Y_0, Z_0, W_0 of the origin o, and the nine direction cosines of the moving axes with respect to the fixed axes. Denote the moving axes by ox, oy, oz, ow; coordinates referred to the fixed axes by X, Y, Z, W, and with respect to the moving axes by x, y, z, w. The moving axes form a moving "tetrahedroid" and the six planes passed through these axes in pairs form a moving "hexahedron." The sixteen direction cosines of the one set of axes referred to the other are given by the following table:

	x	y	z	w
X	α_1	β_1	γ_1	δ_1
Y	α_2	β_2	γ_2	δ_2
Z	α_3	β_3	γ_3	δ_3
W	α_4	β_4	γ_4	δ_4

Consider first the motion of an invariable system, that is, a system whose points do not move relatively to each other, and let x, y, z, w be the coordinates of a point P of the system referred to the moving axes, and X, Y, Z, W be the coordinates of the same point referred to the fixed axes; then

$$\begin{aligned} X &= X_0 + \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 w, \\ Y &= Y_0 + \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 w, \\ Z &= Z_0 + \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3 w, \\ W &= W_0 + \alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4 w. \end{aligned} \tag{1}$$

Differentiating these equations for t, we have the projections of the velocity V of the point P on the fixed axes,

$$\begin{aligned} V_X &= \frac{dX_0}{dt} + x\frac{d\alpha_1}{dt} + y\frac{d\beta_1}{dt} + z\frac{d\gamma_1}{dt} + w\frac{d\delta_1}{dt}, \\ V_Y &= \frac{dY_0}{dt} + x\frac{d\alpha_2}{dt} + y\frac{d\beta_2}{dt} + z\frac{d\gamma_2}{dt} + w\frac{d\delta_2}{dt}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2}$$

the x, y, z, w being constant in an invariable system.

Projecting V_X, V_Y, V_Z, V_W on the axes ox, oy, oz, ow, we have, as the projections V_x, V_y, V_z, V_w of the velocity V on the moving axes,

$$\left.\begin{aligned} V_x &= \alpha_1 V_X + \alpha_2 V_Y + \alpha_3 V_Z + \alpha_4 V_W, \\ V_y &= \beta_1 V_X + \beta_2 V_Y + \beta_3 V_Z + \beta_4 V_W, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{3}$$

These expressions for V_x, V_y, V_z, V_w become simple if we substitute in the second members of (3) the values of V_X, V_Y, V_Z, V_W given in (2), and remember that such sums as

$$\alpha_1 \frac{d\alpha_1}{dt} + \alpha_2 \frac{d\alpha_2}{dt} + \alpha_3 \frac{d\alpha_3}{dt} + \alpha_4 \frac{d\alpha_4}{dt}$$

are zero, and if we use the following abbreviations:

$$\left.\begin{aligned} p_{12} &= \sum \beta_\kappa \frac{d\alpha_\kappa}{dt} = -\sum \alpha_\kappa \frac{d\beta_\kappa}{dt}, \\ p_{13} &= \sum \alpha_\kappa \frac{d\gamma_\kappa}{dt} = -\sum \gamma_\kappa \frac{d\alpha_\kappa}{dt}, \\ p_{23} &= \sum \gamma_\kappa \frac{d\beta_\kappa}{dt} = -\sum \beta_\kappa \frac{d\gamma_\kappa}{dt}, \\ p_{14} &= \sum \delta_\kappa \frac{d\alpha_\kappa}{dt} = -\sum \alpha_\kappa \frac{d\delta_\kappa}{dt}, \\ p_{24} &= \sum \delta_\kappa \frac{d\beta_\kappa}{dt} = -\sum \beta_\kappa \frac{d\delta_\kappa}{dt}, \\ p_{34} &= \sum \delta_\kappa \frac{d\gamma_\kappa}{dt} = -\sum \gamma_\kappa \frac{d\delta_\kappa}{dt}, \end{aligned}\right\} \tag{4}$$

We then find as the expressions of the projections of the velocity on the moving axes,

$$\left.\begin{aligned} V_x &= V_x^0 - p_{14}w + p_{13}z - p_{12}y, \\ V_y &= V_y^0 + p_{12}x - p_{24}w - p_{23}z, \\ V_z &= V_z^0 + p_{23}y - p_{13}x - p_{34}w, \\ V_w &= V_w^0 + p_{34}z + p_{24}y + p_{14}x, \end{aligned}\right\} \tag{5}$$

where V_x^0, V_y^0, V_z^0, V_w^0 denote the projections of the velocity V of the origin of the moving axes on the moving axes,

$$\left.\begin{aligned} V_x^0 &= \alpha_1 \frac{dX_0}{dt} + \alpha_2 \frac{dY_0}{dt} + \alpha_3 \frac{dZ_0}{dt} + \alpha_4 \frac{dW_0}{dt}. \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{6}$$

2. The interpretation of equations (5) is as follows: They show that the velocity V of each point P of the system is the geometric sum of two vectors V and T; the first being the same for all points of the body and equal and parallel to the velocity of the moving origin; and the second, T, differing for different parts of the body and having as its projections on the moving axes

$$\begin{aligned} &-p_{14}w + p_{13}z - p_{12}y\,, \\ &p_{12}x - p_{24}w - p_{23}z\,, \\ &p_{23}y - p_{13}x - p_{34}w, \\ &p_{34}z + p_{24}y + p_{14}x\,. \end{aligned}$$

The vector T is the velocity the point P would possess if the only motion were one of rotation about a plane through the moving origin; the quantities p_{ij} being the projections of the rotation on the six planes of the moving hexahedron, and the four quantities just written being its projections on the four axes.

This rotation may be called the instantaneous rotation, and we say that the velocity of any point in a body is the resultant of a velocity of translation equal to the velocity of any point o of the body, and a velocity of rotation about a plane through o.

Instantaneous plane of rotation.

3. For convenience, I shall speak of any relation between x, y, z, w as the equation of a hypersurface; two relations as the equation of a surface; three relations as the equation of a line. An expression of the form

$$ax + by + cz + dw + e = 0 \tag{7}$$

will be said to represent a hyperplane; two such relations, a plane.

If an invariable system moves in a space of four dimensions, and we wish to find those points in it having at any instant a minimum velocity, we form from (5) the expression

$$V^2 = A^2 + B^2 + C^2 + D^2, \tag{8}$$

where A, B, C, D are the right-hand members of (5), and equate the four derivatives with respect to x, y, z, w to zero. We find

$$\left.\begin{aligned} p_{12}B - p_{13}C + p_{14}D &= 0, \\ -p_{12}A + p_{23}C + p_{24}D &= 0, \\ p_{13}A - p_{23}B + p_{34}D &= 0, \\ -p_{14}A - p_{24}B - p_{34}C &= 0, \end{aligned}\right\} \tag{9}$$

which, solved for x, y, z, w, give the points sought. To get the minimum velocity itself, we solve (9) for x, y, z, w and substitute their values in (8). The system (9) being linear and homogeneous in A, B, C, D, we will divide each equation in it through by D and solve the first three of the resulting equations for the ratios of A, B, C to D. This supposes that all the quantities A, B, C, D are not zero; that is, that the velocity V is not zero. We find

$$\frac{A}{D} = \begin{vmatrix} -p_{14} & p_{12} & -p_{13} \\ -p_{24} & 0 & p_{23} \\ -p_{34} & -p_{23} & 0 \end{vmatrix} \div \begin{vmatrix} 0 & p_{12} & -p_{13} \\ -p_{12} & 0 & p_{23} \\ p_{13} & -p_{23} & 0 \end{vmatrix}; \tag{10}$$

a determinant whose denominator is a skew symmetric determinant of odd order and, therefore, zero. Then the numerator

$$\begin{vmatrix} -p_{14} & p_{12} & -p_{13} \\ -p_{24} & 0 & p_{23} \\ -p_{34} & -p_{23} & 0 \end{vmatrix} = -(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}) \tag{11}$$

must vanish; that is, *in order that there be in a moving system a set of points with minimum velocity not zero, the relation*

$$(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}) = 0 \tag{12}$$

must be satisfied between the six quantities p_{ij}.

The determinant in (11) being the determinant of the first three equations (9) divided by D and it being zero, and the equations being non-homogeneous, it follows that these equations are not independent and (9) *represents a plane*, the locus of all points whose velocity is minimum, *called the central plane of motion.* The motion may be looked on as made up of a gliding along the central plane and a rotation about it. No matter which three equations (9) are taken, their determinant is zero.

If the velocity at any instant *is* zero, that is, if the motion reduce entirely to a rotation, we must have

$$V^2 = A^2 + B^2 + C^2 + D^2 = 0;$$

therefore $$A = B = C = D = 0,$$

or

$$\left.\begin{aligned} V_x^0 - p_{14}w + p_{13}z - p_{12}y &= 0, \\ V_y^0 + p_{12}x - p_{24}w - p_{23}z &= 0, \\ V_z^0 + p_{23}y - p_{13}x - p_{34}w &= 0, \\ V_w^0 + p_{34}z + p_{24}y + p_{14}x &= 0. \end{aligned}\right\} \tag{13}$$

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These equations (13) have as their determinant

$$\begin{vmatrix} 0 & p_{12} & -p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{vmatrix},$$

which, being a skew-symmetric determinant of even order, may be written

$$(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23})^2.$$

The expression in parenthesis is the first member of (12) and if we take

$$(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}) = 0,$$

the four equations (13) reduce to two which *represent a plane called the instantaneous plane of rotation*, it being the locus of all points having at the instant considered a velocity equal to zero.

When the moving system has one point fixed we may take the fixed point for the origin of both the fixed and the moving axes; the equations (13) reduce to

$$\left.\begin{aligned} -p_{14}w + p_{13}z - p_{12}y &= 0, \\ p_{12}x - p_{24}w - p_{23}z &= 0, \\ p_{23}y - p_{13}x - p_{34}w &= 0, \\ p_{34}z + p_{24}y + p_{14}x &= 0. \end{aligned}\right\} \tag{14}$$

a set of four linear, homogeneous equations with four unknowns. If these are to be satisfied by a set of values of x, y, z, w not all zero, it is necessary that their determinant, which is exactly the determinant written after equations (13) vanish. In this case the four equations reduce to two and we have an instantaneous plane passing through the fixed point.

If in equations (13) we make

$$V_w^0 = w = p_{14} = p_{24} = p_{34} = 0,$$

the resulting equations

$$\begin{aligned} V_x^0 + p_{13}z - p_{12}y &= 0, \\ V_y^0 + p_{12}x - p_{23}z &= 0, \\ V_z^0 + p_{23}y - p_{13}x &= 0 \end{aligned}$$

are the equations of the instantaneous axis of rotation of an invariable system moving in a space of four dimensions.

Velocities in any system.

4. Suppose now that the moving system is not an invariable system, but that its points are free to move among themselves. Then, instead of treating x, y, z, w as constants in differentiating equations (1), we treat them at functions of the time t, and equations (5) become

$$\begin{aligned} V_x &= V_x^0 - p_{14}w + p_{13}z - p_{12}y + \frac{dx}{dt}, \\ V_y &= V_y^0 + p_{12}x - p_{24}w - p_{23}z + \frac{dy}{dt}, \\ V_z &= V_z^0 + p_{23}y - p_{13}x - p_{34}w + \frac{dz}{dt}, \\ V_w &= V_w^0 + p_{34}z + p_{24}y + p_{14}x + \frac{dw}{dt}, \end{aligned} \tag{15}$$

which are the projections of the velocity of a moving point in a moving system upon a set of axes invariably connected with the moving system.

Expressions for the rotations in terms of the direction cosines.

5. As in case of rotations in a space of three dimensions (see Darboux, I, p. 4), the six quantities p_{ij} may be expressed in terms of the sixteen direction cosines $\alpha_1, \ldots, \delta_4$ and their derivatives with respect to the time. (See *American Journal of Mathematics*, vol. 20, p. 138.) Suppose, constructed through a fixed point of space, a tetrahedroid T' with the same arrangement of axes as in the one T fixed in the moving system, and let the axes of T' always remain parallel to those of T. Then, as far as the rotation is concerned, the motion of T' is the same as that of T, and the motion of T' will be given by equations (15) when the terms $V_x^0, \ldots, V_w^0$ are made zero. Take a point on the fixed axis OX at a distance of unity from the origin; its coordinates referred to T' are $\alpha_1, \beta_1, \gamma_1, \delta_1$. Its velocity being zero, equations (14) give

$$\begin{aligned} \frac{d\alpha_1}{dt} &= p_{12}\beta_1 - p_{13}\gamma_1 + p_{14}\delta_1, \\ \frac{d\beta_1}{dt} &= -p_{12}\alpha_1 + p_{23}\gamma_1 + p_{24}\delta_1, \\ \frac{d\gamma_1}{dt} &= p_{13}\alpha_1 - p_{23}\beta_1 + p_{34}\delta_1, \\ \frac{d\delta_1}{dt} &= -p_{14}\alpha_1 - p_{24}\beta_1 - p_{34}\gamma_1. \end{aligned} \tag{16}$$

Similarly we may get three other sets of equations like the above, except that the subscript 1 is replaced by 2, 3, 4 respectively.

Making, successively, the subscript 1, 2, 3, 4 in the first equation (16), multiplying the resulting equations respectively by β_1, β_2, β_3, β_4, and adding, we find

$$\sum_1^4 \beta_\kappa \frac{d\alpha_\kappa}{dt} = \sum p_{12}\beta_\kappa^2 - \sum p_{13}\beta_\kappa\gamma_\kappa - \sum p_{14}\beta_\kappa\delta_\kappa,$$

or
$$p_{12}dt = \Sigma\,\beta_\kappa d\alpha_\kappa.$$

Similarly
$$p_{12}dt = \Sigma\beta d\alpha = -\Sigma\alpha d\beta\,; \quad p_{23}dt = \Sigma\gamma d\beta = -\Sigma\beta d\gamma,$$
$$p_{13}dt = \Sigma\alpha d\gamma = -\Sigma\gamma d\alpha\,; \quad p_{24}dt = \Sigma\delta d\beta = -\Sigma\beta d\delta,$$
$$p_{14}dt = \Sigma\delta d\alpha = -\Sigma\alpha d\delta\,; \quad p_{34}dt = \Sigma\delta d\gamma = -\Sigma\gamma d\delta,$$

a set of equations expressing the rotations in terms of the sixteen direction cosines and their derivatives with respect to the time. They are analogous to those for a space of three dimensions. (Darboux, I, p. 5.)

APPLICATION TO CURVES OF TRIPLE CURVATURE.

The principal directions.

6. When a line is such that five consecutive points on it are not in general situated in the same ordinary space, it is said to be *a line of triple curvature.* Two consecutive points determine the *tangent* at any point; three consecutive points determine the *ordinary osculating plane* and four consecutive points determine the *osculating hyperplane.* The straight line situated in the osculating plane and perpendicular to the tangent is called the *principal normal* at the point M of tangency. The straight line contained in the osculating hyperplane and perpendicular to the tangent and the principal normal is called the *principal binormal* at M; and a fourth line perpendicular to the three just defined, and, consequently, to the osculating hyperplane also, is called the *trinormal* at M. The binormal and trinormal are so called because they are respectively perpendicular to two and three consecutive tangents. The four lines just defined determine what we call the *principal directions* at any point M of a curve of triple curvature, and the tetrahedroid T formed by them at M is called the *principal tetrahedroid* at the point. T is fixed to the curve, and as M moves along L the tetrahedroid T moves in space. Its motion is given by the kinematical equations which precede, and by its use formulæ may be found for curves L corresponding to the Serret-Frenet formulæ for curves of double curvature.

The three curvatures.

7. The limit of the ratio of the angle of two consecutive tangents to the arc of the curve included between their points of contact is the first curvature at the point considered, and is written

$$\lim_{s \doteq 0} \left[\frac{d\phi_1}{ds}\right] = \frac{1}{\rho_1} \equiv \text{first curvature.}$$

If, through the origin of the fixed axes, we draw lines k of length 1 parallel to the tangents of the curve L, their extremities will trace out a spherical curve Γ on a sphere of unit radius having its center at o, and the coordinates of the extremity of any one of these lines are the direction cosines of the corresponding tangent to L. The angle between two consecutive lines k is the angle between two consecutive tangents of L, and is measured by the arc of Γ included between its sides. Then, by reason of the above definition of the first curvature, we may write

$$\frac{1}{\rho_1} = \sqrt{\frac{d\alpha_1^2}{ds^2} + \frac{d\alpha_2^2}{ds^2} + \frac{d\alpha_3^2}{ds^2} + \frac{d\alpha_4^2}{ds^2}}\,.$$

The limit of the ratio of two consecutive osculating planes to the arc between the point of contact is the second curvature of the curve considered:

$$\lim_{s \doteq 0} \left[\frac{d\phi_2}{ds}\right] = \frac{1}{\rho_2} = \text{second curvature.}$$

Making use of a method similar to the one just used for getting the second expression for the first curvature, we may write

$$\frac{1}{\rho_2} = \sqrt{\frac{d\gamma_1^2}{ds^2} + \frac{d\gamma_2^2}{ds^2} + \frac{d\gamma_3^2}{ds^2} + \frac{d\gamma_4^2}{ds^2}}\,.$$

The limit of the ratio of the angle of two consecutive trinormals to the arc ds is the third curvature of the point considered:

$$\lim_{s \doteq 0} \left[\frac{d\phi_3}{ds}\right] = \frac{1}{\rho_3} = \text{third curvature.}$$

We also have

$$\frac{1}{\rho_3} = \sqrt{\frac{d\delta_1^2}{ds^2} + \frac{d\delta_2^2}{ds^2} + \frac{d\delta_3^2}{ds^2} + \frac{d\delta_4^2}{ds^2}}\,.$$

Interpretation of the quantities p_{ij}.

8. If, at any ordinary point M of a curve L, we form the tetrahedroid T and take the tangent as the axis of x, the principal normal as the axis of y, the principal binormal as the axis of z and the trinormal as the axis of w, we have a system of moving rectangular axes, with the origin at M, to which points on the curve may be referred. Let x, y, z, w be the coordinates of any point referred to these moving axes, and X, Y, Z, W the coordinates of the same point referred to a set of rectangular axes, fixed in space, having an arbitrary point o as origin. Take the arc s of the curve as the independent variable, or, what is the same thing, suppose

$$\frac{ds}{dt} = 1.$$

Then $$V_x^0 = 1, \quad V_y^0 = V_z^0 = V_w^0 = 0;$$

and also,

$$\alpha_1 = \frac{dx}{ds}, \quad \alpha_2 = \frac{dy}{ds}, \quad \alpha_3 = \frac{dz}{ds}, \quad \alpha_4 = \frac{dw}{ds}, \tag{17}$$

From equations (16) we find

$$\left.\begin{aligned} d\alpha_\kappa &= (\quad p_{12}\beta_\kappa - p_{13}\gamma_\kappa + p_{14}\delta_\kappa)\,ds, \\ d\beta_\kappa &= (-p_{12}\alpha_\kappa + p_{23}\gamma_\kappa + p_{24}\delta_\kappa)\,ds, \\ d\gamma_\kappa &= (\quad p_{13}\alpha_\kappa - p_{23}\beta_\kappa + p_{34}\delta_\kappa)\,ds, \\ d\delta_\kappa &= (-p_{14}\alpha_\kappa - p_{24}\beta_\kappa - p_{34}\gamma_\kappa)\,ds. \end{aligned}\right\} \tag{18}$$

Since the principal binormal, whose direction cosines are $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, is perpendicular to two consecutive tangents, whose direction cosines are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3, \alpha_4 + d\alpha_4$, it follows that

$$\sum_1^4 (\alpha_\kappa \gamma_\kappa) = 0 \quad \text{and} \quad \sum \gamma_\kappa (\alpha_\kappa + d\alpha_\kappa) = 0.$$

The first of these is true on account of the orthogonality of the axes; and, if we remember that

$$\Sigma \gamma_\kappa d\alpha_\kappa = -p_{13} ds,$$

the second will give the condition

$$\Sigma \gamma_\kappa d\alpha_\kappa = -p_{13} ds = 0,$$

and, therefore, $$p_{13} = 0.$$

This corresponds to the condition $q = 0$ for ordinary space.

If, in the same way, we express the condition that the trinormal is perpendicular to three consecutive tangents, we find the additional necessary relation

$$\Sigma\delta_\kappa(\alpha_\kappa + d\alpha_\kappa) = 0, \quad \text{or} \quad \Sigma\delta_\kappa d\alpha_\kappa = p_{14}ds = 0,$$

and, therefore,

$$p_{14} = 0.$$

Two consecutive osculating hyperplanes of a curve of triple curvature intersect in an osculating plane; then the principal normal to L at M, since it lies in the osculating plane, lies also in both osculating hyperplanes and is therefore perpendicular to two consecutive trinormals. Expressing this fact we have

$$\Sigma\beta_\kappa(\delta_\kappa + d\delta_\kappa) = 0, \quad \text{or} \quad \Sigma\beta_\kappa d\delta_\kappa = -p_{24}ds = 0,$$

which requires

$$p_{24} = 0.$$

Of the six rotations p_{ij} three are zero, as we have seen, when the tetrahedroid T is connected with the curve as above described. In order to interpret the remaining three it is necessary to use the definitions given in (7) of the three curvatures of a line L.

9. Construct through some fixed point a tetrahedroid T_1 whose axes are parallel to those of T. We get in this way a tetrahedroid whose motion, so far as rotation is concerned, is exactly that of T. A point at the distance 1 from the origin on the x-axis of T_1 has a velocity whose projections (see equations (5)) are $0, p_{12}, 0, 0$; that is to say, this point describes a path $p_{12}\,ds$; or, what is the same thing, the tangent to the curve turns through an angle $p_{12}\,ds$ while the point of contact describes the arc ds. Consequently

$$\text{the first curvature} = \frac{1}{\rho_1} = \frac{p_{12}ds}{ds} = p_{12}.$$

The projections of the velocity of a point on the principal binormal at a distance 1 from the origin are $0, -p_{23}, 0, p_{34}$. Therefore,

$$\text{the second curvature} = \frac{1}{\rho_2} = \frac{\sqrt{p_{34}^2 + (-p_{23})^2}\,.\,ds}{ds},$$

$$\frac{1}{\rho_2} = \sqrt{p_{34}^2 + (-p_{23})^2}.$$

A point at the distance 1 from the origin and on the trinormal has a velocity whose projections are $0, 0, -p_{34}, 0$; therefore,

$$\text{the third curvature} = \frac{1}{\rho_3} = \frac{-p_{34}ds}{ds} = -p_{34}.$$

These last three formulæ give us the desired interpretation for the quantities p_{12}, p_{23}, p_{34};

$$\left.\begin{aligned} p_{12} &= \frac{1}{\rho_1}, \\ p_{23} &= -\sqrt{\frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}} \equiv \frac{1}{H}, \\ p_{34} &= -\frac{1}{\rho_3}, \end{aligned}\right\} \tag{19}$$

When $\rho_3 = \infty$ these reduce immediately to the corresponding formulæ for ordinary space.

"*Serret-Frenet*" *Formulæ.*

10. If we introduce the values given by (19) into (18) and also use $p_{13} = p_{14} = p_{24} = 0$, we find

$$\left.\begin{aligned} \frac{d\alpha_\kappa}{ds} &= \frac{\beta_\kappa}{\rho_1}, \\ \frac{d\beta_\kappa}{ds} &= -\frac{\alpha_\kappa}{\rho_1} - \frac{\gamma_\kappa}{H}, \\ \frac{d\gamma_\kappa}{ds} &= -\frac{\delta_\kappa}{\rho_3} + \frac{\beta_\kappa}{H}, \\ \frac{d\delta_\kappa}{ds} &= \frac{\gamma_\kappa}{\rho_3}, \end{aligned}\right\} \tag{20}$$

a set of equations which express the derivatives with respect to the arc of the sixteen direction cosines as linear functions of the cosines themselves; the coefficients being functions of the three curvatures of the curve considered. Formulæ (20) are to curves of triple curvature what the Serret-Frenet formulæ are to curves of double curvature.

From equations (20) follow immediately on squaring and adding

$$\left.\begin{aligned} \frac{d\alpha_\kappa^2}{ds^2} &= \frac{1}{\rho_1^2}, & \frac{d\gamma_\kappa^2}{ds^2} &= \frac{1}{\rho_2^2}, \\ \frac{d\beta_\kappa^2}{ds^2} &= \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}, & \frac{d\delta_\kappa^2}{ds^2} &= \frac{1}{\rho_3^2}, \end{aligned}\right\} \tag{21}$$

the first, third and fourth of which we have already obtained.

Spherical Indicatrix.

11. Suppose, through a fixed point, we construct a tetrahedroid with axes parallel to those of the principal tetrahedroid at a point M of a curve L, and at a distance of 1 from the origin on each of the axes parallel to the tangent, principal binormal and trinormal, choose points. These points will describe spherical curves on a sphere of radius unity whose center is the fixed point. We will call the curves σ, τ, k so described, the spherical indicatrix of the tangent, the spherical indicatrix of the principal binormal and the spherical indicatrix of the trinormal respectively.

By means of certain properties of these curves σ, τ, k obtained by using the kinematical equations (15), we can define the positive directions of the principal normal, principal binormal and trinormal, having given a positive direction on the tangent. For any point on σ equations (15) and (19), together with $p_{13} = p_{14} = p_{24} = 0$, give

$$V_x = 0, \quad V_y = \frac{1}{\rho_1}, \quad V_z = 0, \quad V_w = 0.$$

Then *the tangent to* σ *is parallel to the principal normal to* L, *and the velocity of the point which traces out* σ *is exactly the first curvature.* We choose, then, as the positive sense on the principal normal, the sense of the tangent to σ.

Imagine a man standing with his feet at the point M of L and the principal binormal passing through him lengthwise; the direction from his feet to his head will be called the positive direction of the principal binormal if he sees the x-axis become coincident with the y-axis after a rotation of 90° from left to right about the principal binormal.

If the line through the fixed point and parallel to the trinormal extends in both directions a distance unity from the fixed point, there will be two spherical curves k and k' traced on the unit sphere by points whose velocity is given by

$$V_x = 0, \; V_y = 0, \; V_z = \frac{1}{\rho_3}, \; V_w = 0.$$

The velocity of the tracing points is therefore $\frac{1}{\rho_3}$ and *the tangents to* k *and* k' *are parallel to the principal binormal* of L. We shall speak of that one of the curves k and k' as the spherical indicatrix of the trinormal which has as its sense the positive sense along the principal binormal. Then the positive sense along the trinormal, will be from the fixed point to the spherical indicatrix of the trinormal.

Then, given a positive sense along the tangent to a curve L at a point M, the positive directions along the three other principal lines are determined.

For any point on τ the equations (15) and (19) give

$$V_x = 0,\ V_y = -p_{23} = \sqrt{\frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}},\ V_z = 0,\ V_w = p_{34} = -\frac{1}{\rho_3},$$

which shows that the motion is in the plane yw and the velocity of the tracing point is the second curvature.

Similarly the point at a distance of 1 along the principal normal traces a curve Γ with a velocity given by

$$V^2 = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{1}{\rho_3^2},$$

What precedes shows that *the curves σ, τ, k are traced by points whose velocities are respectively the first, second and third curvatures of the curve considered, while the square of the velocity of the point tracing Γ is the sum of the squares of the velocity along σ and τ minus the square of the velocity along k.*

Principal directions and three curvatures at a point in terms of the coordinates of the point.

12. The osculating hyperplane at any point M of a curve of triple curvature may be defined as a hyperplane such that the distance to it from any point of the line infinitely near M is an infinitesimal of at least the fourth order. Being so defined we can get its equation

$$A(X-x) + B(Y-y) + C(Z-z) + D(W-w) = 0. \quad A = \begin{vmatrix} y' & z' & w' \\ y'' & z'' & w'' \\ y''' & z''' & w''' \end{vmatrix} \tag{22}$$

by a method exactly similar to that used in getting the equation of the osculating plane of a curve of double curvature.

When the arc is taken as the independent variable

$$\alpha_1 = \frac{dx}{ds} = x',\ \alpha_2 = y',\ \alpha_3 = z',\ \alpha_4 = w'. \tag{23}$$

Also if M be a point on L and m be the corresponding point on the spherical indicatrix of the tangents to L, then, since the tangent at m is parallel to the

principal normal at M, the cosines of the principal normal to L at M are given by

$$\beta_1 = \frac{d\alpha_1}{d\sigma}, \ldots, \beta_4 = \frac{d\alpha_4}{d\sigma},$$

where σ is the arc of the spherical indicatrix. Remembering that $\frac{d\sigma}{ds} = \frac{1}{\rho_1}$, we have

$$\beta_1 = \rho_1 \frac{d\alpha_1}{ds} = \rho_1 x'', \ldots \beta_4 = \rho_1 w''. \tag{24}$$

The direction cosines of the trinormal being proportional to those of the perpendicular to the osculating plane, we have

$$\delta_1 = \frac{A}{\Delta}, \ldots, \delta_4 = \frac{D}{\Delta}, \ \Delta^2 = A^2 + B^2 + C^2 + D^2. \tag{25}$$

If m' be the point on the spherical indicatrix of the trinormals corresponding to M, then, since the tangent at m' is parallel to the principal binormal of L,

$$\gamma_1 = \frac{d\delta_1}{d\sigma}, \ldots, \gamma_4 = \frac{d\delta_4}{d\sigma},$$

$d\sigma$ being the arc of the spherical indicatrix. Since $\frac{d\sigma}{ds} = \frac{1}{\rho_3}$

$$\gamma_1 = \rho_3 \frac{d\delta_1}{ds}, \ldots, \gamma_4 = \rho_3 \frac{d\delta_4}{ds}. \tag{26}$$

If we notice that the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = 1,$$

each element is exactly equal to its minor, we will have

$$\gamma_1 = \begin{vmatrix} \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \\ \alpha_4 & \beta_4 & \delta_4 \end{vmatrix}, \text{ etc,} \tag{27}$$

Squaring and adding equations (24) we find an expression for the first curvature in terms of the coordinates of the point;

$$\frac{1}{\rho_1^2} = x''^2 + y''^2 + z''^2 + w''^2. \tag{28}$$

Squaring and adding (26), we find again

$$\frac{1}{\rho_3^2} = \left(\frac{d\delta_1}{ds}\right)^2 + \left(\frac{d\delta_2}{ds}\right)^2 + \left(\frac{d\delta_3}{ds}\right)^2 + \left(\frac{d\delta_4}{ds}\right)^2, \tag{29}$$

which, combined with (25) gives the third curvature in terms of the coordinates of the point.

The expression for the second curvature in terms of the coordinates of the point may be obtained by using

$$\frac{1}{\rho_2^2} = \frac{d\gamma_1^2 + d\gamma_2^2 + d\gamma_3^2 + d\gamma_4^2}{ds^2},$$

together with (26), (25) and (29).

The three curvatures being found in terms of the coordinates of M, we can, by substitution in (23), (24), (27) and (25), find $\alpha_1, \ldots, \delta_4$ in terms of the same coordinates.

Position of a curve L.

13. The positive sense on the principal binormal being defined as above, the center of first curvature is on the concave side of the curve. For, if P and P_1 are two infinitely near points on the curve, the projection of the chord PP_1 on the principal normal is

$$K = \beta_1(x_1 - x) + \beta_2(y_1 - y) + \beta_3(z_1 - z) + \beta_4(w_1 - w).$$

If the equations of the curve are

$$x = f_1(s), \quad y = f_2(s), \ldots,$$

then $$x_1 = f_1(s+h), \quad y_1 = f_2(s+h), \ldots,$$

and $$x_1 = f_1(s) + hf_1'(s) + \frac{h^2}{2} f_1''(s) + \ldots.$$

But $$f_1(s) = x, \quad f_1'(s) = \alpha_1, \quad f_1''(s) = \frac{d\alpha_1}{ds} = \frac{\beta_1}{\rho_1}, \ldots,$$

then $$x_1 - x = h\alpha_1 + \frac{h^2\beta}{2\rho_1} + \frac{h^3}{6}(\beta_1\phi' - \alpha_1\phi^2 - \phi\psi\gamma_1) + \frac{h^4}{24}(\ldots) + \ldots.$$

where $$\phi = \frac{1}{\rho_2}, \quad \psi = \frac{1}{H}, \quad \chi = \frac{1}{\rho_3}.$$

Similar expressions may be obtained for $y_1 - y$, $z_1 - z$, $w_1 - w$ by advancing the

subscripts on the direction cosines. Substituting in the expression for K, we have

$$K^2 = \frac{h^2}{2\rho_1} + \ldots.$$

which is positive, whether h be $-$ or $+$, provided it be infinitesimal. The projection K being positive, the center of curvature must be on the concave side of the curve. *The curve is entirely on one side of its tangent.*

The projection of PP_1 on the principal binormal is given by

$$\begin{aligned} K_1 &= \gamma_1(x_1 - x) + \gamma_2(y_1 - y) + \ldots. \\ &= -\frac{h^3\phi\psi}{6} + \ldots.; \end{aligned}$$

that is, according as h is $+$ or $-$, the projection PP_1 on the principal binormal is $-$ or $+$. *The curve, therefore, crosses its osculating plane.*

The projection K_2 on the trinormal is given by

$$\begin{aligned} K_2 &= \delta_1(x_1 - x) + \delta_2(y_1 - y) + \ldots. \\ &= \frac{h^4\phi\psi\chi}{24} + \ldots.; \end{aligned}$$

that is, whether h be $+$ or $-$, *the curve is always on the same side of its osculating hyperplane.* Conf. Lovett, *Amer. Jour.*, vol. XXII, p. 226 et seq.

Coordinates of a point in terms of the arc.

14. To express the coordinates of any point of a curve L in terms of the arc, we let O be a point of L and M a point near it. Take O as the origin and construct the usual tetrahedroid. The coordinates of M are

$$x = f_1(s), \quad y = f_2(s), \ldots.,$$

which, developed by McLaurin's formula, give

$$x = sf_1'(0) + \frac{s^3}{2}f_1''(0) + \frac{s^2}{6}f_1'''(0) + \ldots$$

Substituting and using the abbreviations Φ, Ψ, X as in the previous section, we find

$$\left.\begin{aligned} x &= s - \alpha_1 \phi^2 s^3 \quad + \ldots\ldots, \\ y &= \frac{\phi}{2} s^2 + \frac{\phi'}{6} s^3 + \ldots\ldots, \\ z &= -\frac{\phi\psi}{6} s^3 \quad + \ldots\ldots, \\ w &= \frac{\phi\psi\chi}{24} s^4 \quad + \ldots\ldots \end{aligned}\right\} \qquad (30)$$

it being understood that the values of Φ, Ψ, X are taken for $s = 0$. These formulæ also make it evident that the curve does not, in general, cross its osculating hyperplane.

It is easy to show by means of (30) that the difference between an infinitesimal arc and its chord is of the third order with respect to the arc.

The osculating hypersphere.

15. An equation of the form

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + (w - w_1)^2 = r^2$$

will be said to represent a hypersphere because of the similarity of its form to that of the equation of a sphere in ordinary space. The conditions for contact of order n between a curve of triple curvature and a hypersurface are like those for contact of a curve of double curvature and a surface in ordinary space, if the definition of contact of order n is:

When a curve L and a hypersurface S have in common a simple point P we say that they have at that point a contact of order n if, for every point Q infinitely near P on L, we can make correspond on S a point Q' such that the distance QQ' be infinitesimal of order $n+1$ with respect to the chord PQ.

The hypersphere which, at a point $M(x, y, z, w)$ of a curve L, has a contact of highest order possible with L, is called the osculating hypersphere of L at M. Since the equation of the hypersphere contains five parameters, the contact must be of the fourth order if the hypersphere is to osculate the curve, and if we call X, Y, Z, W the coordinates of the center of the osculating hypersphere and R its radius, we have

$$\begin{aligned} \phi(s) &= \Sigma (x - X)^2 - R^2 = 0, \\ \tfrac{1}{2}\phi'(s) &= \Sigma (x - X)\alpha_1 \quad = 0, \\ \phi''(s) &= \phi'''(s) = \phi^{IV}(s) \quad = 0. \end{aligned}$$

The second of these is the equation of the normal hyperplane at M. If we differentiate the second equation and use the first equation (20); differentiate again and use the second of (20), and, finally, differentiate again and use the third equation (20), we have, for the determination of the radius R, the equations

$$\left.\begin{aligned}
(x-X)\alpha_1+(y-Y)\alpha_2+(z-Z)\alpha_3+(w-W)\alpha_4 &= 0,\\
(x-X)\beta_1+(y-Y)\beta_2+(z-Z)\beta_3+(w-W)\beta_4 &= -\rho_1,\\
(x-X)\gamma_1+(y-Y)\gamma_2+(z-Z)\gamma_3+(w-W)\gamma_4 &= H\frac{d\rho_1}{ds},\\
(x-X)\delta_1+(y-Y)\delta_2+(z-Z)\delta_3+(w-W)\delta_4 & \\
&= -\left(\frac{\rho_1\rho_3}{H}+\rho_3\frac{d}{ds}H\frac{d\rho_1}{ds}\right).
\end{aligned}\right\} \quad (31)$$

Squaring and adding these four equations we find

$$\begin{aligned}
(x-X)^2+(y-Y)^2+(z-Z)^2+(w-W)^2 &= R^2\\
&= \rho_1^2+H^2\frac{d\rho_1^2}{ds^2}+\left(\frac{\rho_1\rho_3}{H}+\rho_3\frac{d}{ds}H\frac{d\rho_1}{ds}\right)^2, \qquad (32)
\end{aligned}$$

which is *the expression sought for the radius of the osculating hypersphere at any point of a curve of triple curvature.* The first two terms form the expression for the radius of the osculating sphere at a point of a curve of double curvature.

*Locus of centres of osculating hyperspheres.**

16, The consecutive normal hyperplanes of a curve L of triple curvature envelop a curved hypersurface, S_4, which is called the *polar hypersurface* of L. Two consecutive normal hyperplanes have in common a plane called the *polar plane* of L. These polar planes envelop a developable surface S_3 called the *surface of regression* of S_4. Three consecutive normal hyperplanes have in common a right line called the *polar line* of L, and these polar lines envelop a curve called the *edge of regression of* S_3 or the *line of regression of* S_4. Then, with these conventions, what precedes shows that *the center of the osculating hypersphere is at the point where the polar line touches its envelope, and its locus is the edge of regression of the polar hypersurface.*

The coordinates of the center of the osculating hypersphere may be found by

* Pirondini, "Giornali di Mathematiche"; loc. cit.

solving equations (31) for $x - X$, $y - Y$, $z - Z$, $w - W$, a process which gives

$$\left.\begin{aligned} X &= x + \beta_1\rho_1 - \gamma_1 H\frac{d\rho_1}{ds} + \delta_1 K, \quad Z = z + \beta_3\rho_1 - \gamma_3 H\frac{d\rho_1}{ds} + \delta_3 K, \\ Y &= y + \beta_2\rho_1 - \gamma_2 H\frac{d\rho_1}{ds} + \delta_2 K, \quad W = w + \beta_4\rho_1 - \gamma_4 H\frac{d\rho_1}{ds} + \delta_4 K, \\ K &= \frac{\rho_1\rho_3}{H} + H\rho_3\frac{d^2\rho_1}{ds^2} + \rho_3\frac{d\rho_1}{ds}\frac{dH}{ds}. \end{aligned}\right\} \quad (33)$$

If equations (33) be differentiated and the results simplified by the use of (20), the resulting formulae will give the direction cosines of the tangent to the curve traced out by the center of the osculating hypersphere;

$$\frac{dX}{ds} = \delta_1\left(\frac{H}{\rho_3}\frac{d\rho_1}{ds} + \frac{dK}{ds}\right), \quad \frac{dY}{ds} = \delta_2\left(\frac{H}{\rho_3}\frac{d\rho_1}{ds} + \frac{dK}{ds}\right), \ldots . \quad (34)$$

The second members of these equations containing δ_1, δ_2, δ_3, δ_4 as factors, it follows that *the tangent to the locus of the centers of the osculating hyperspheres of a curve L of triple curvature is parallel to the trinormal at the corresponding point of L.*

If L be a curve of triple curvature and L_0, be the locus of the centers of its osculating hyperspheres, the osculating hyperplane of L is parallel to the normal hyperplane of L_0, since the trinormal of L is parallel to the tangent of L_0. Also, the osculating hyperplane of L_0 having the polar line of L as its characteristic coincides with the normal hyperplane of L and, consequently, the trinormal of L, is parallel to the tangent of L. The principal binormal of L_0 lies in the osculating hyperplane of L_0 perpendicular to two consecutive tangents of L_0; that is, it lies in the normal hyperplane of L perpendicular to two consecutive trinormals of L. Therefore it is parallel to the principal normal. It follows immediately that the principal normal of L_0 is parallel to the principal binormal of L. We have then the theorem;

The tangent, principal normal, principal binormal and trinormal of a curve L of triple curvature are respectively parallel to the trinormal, principal binormal, principal normal and tangent of the locus L_0 of the centers of the osculating hyperspheres of L.

17. The tangent to L_0 at any point being parallel to the trinormal of L at the corresponding point, and the trinormal of L_0 being parallel to the tangent of L, it follows that the angle of two tangents of L_0 is equal to the angle of two

osculating hyperplanes of L; and the angle of two osculating hyperplanes of L_0 is equal to the angle of two tangents to L. We have therefore

$$\frac{\text{angle of two tangents of } L_0}{ds_0} = \frac{1}{\rho_1^0},$$

$$\frac{\text{angle of two osculating hyperplanes of } L}{ds} = \frac{1}{\rho_3},$$

and, consequently,

$$\frac{ds_0}{\rho_1^0} = \frac{ds}{\rho_3}.$$

In like manner,

$$\frac{\text{angle of two osculating hyperplanes of } L_0}{ds_0} = \frac{1}{\rho_3^0},$$

$$\frac{\text{angle of two tangents of } L}{ds} = \frac{1}{\rho_1},$$

whence,

$$\frac{ds_0}{\rho_3^0} = \frac{ds}{\rho_1},$$

where ds^0, ρ_1^0, ρ_3^0 denote the arc, radius of first curvature and radius of third curvature of L_0. Comparing these results, we find immediately the relation

$$\rho_1\rho_1^0 = \rho_3\rho_3^0.$$

We can also find ρ_1^0 and ρ_3^0; for, from (26), we have

$$\frac{ds_0}{ds} = \frac{H}{\rho_3}\frac{d\rho_1}{ds} + \frac{dK}{ds} \equiv V.$$

Consequently, *given the three curvatures of a curve of triple curvature, we can find immediately the first and third curvatures of the locus of the centers of its osculating hyperspheres by means of the equations*

$$\rho_1^0 = \rho_3 V, \quad \rho_3^0 = \rho_1 V;$$

V being given by the equation above.

18. *The osculating hypersphere and the osculating hyperplane at a point M of L intersect in an ordinary sphere which is the ordinary osculating sphere at M, and is the characteristic of the osculating hypersphere at M.*

For the characteristic of the osculating hypersphere at M is given by

$$S(x-X)^2 - R^2 = 0,$$
$$S(x-X)\frac{dX}{ds} + R\frac{dR}{ds} = 0,$$

the latter being the equation of the hyperplane determining the characteristic. By using (34), this last equation may be written

$$[S(x-X)\delta_1]\Big[\frac{H}{\rho_3}\frac{d\rho_1}{ds} + \frac{dK}{ds}\Big] + R\frac{dR}{ds} = 0, \tag{35}$$

which shows that the hyperplane determining the characteristic is parallel to the osculating hyperplane. It also passes through the point M; for, substituting from (33) the values of X, Y, Z, W, we find

$$R\Big[\frac{H}{\rho_1}\frac{d\rho_1}{ds} + \frac{dK}{ds}\Big] - R\frac{dR}{ds} = 0,$$

which is seen to be verified if we differentiate (32).

Hyperspherical curves.

19. Any curve of triple curvature which is situated on a hypersphere is called a hyperspherical curve. Since the osculating hypersphere of such a curve is the hypersphere on which it lies, we have necessarily at every point of a hyperspherical curve

$$\rho_1^2 + H^2\frac{d\rho_1^2}{ds^2} + K^2 = \text{const.} \tag{36}$$

Conversely, if at every point of a curve of triple curvature the radius of the osculating hypersphere is constant, the curve is a hyperspherical curve. For, noticing that

$$\frac{K}{\rho_3} - \frac{d}{ds}\Big(H\frac{d\rho_1}{ds}\Big) - \frac{\rho_1}{H} = 0,$$

and, therefore,

$$\frac{H}{\rho_3}\frac{d\rho_1}{ds} = \frac{H}{K}\frac{d}{ds}\Big(H\frac{d\rho_1}{ds}\Big) + \frac{\rho_1}{K}\frac{d\rho_1}{ds},$$

we can get from equations (26),

$$\frac{1}{\delta_1}\frac{dX}{ds}=\frac{1}{\delta_2}\frac{dY}{ds}=\frac{1}{\delta_3}\frac{dZ}{ds}=\frac{1}{\delta_4}\frac{dW}{ds}$$
$$=\frac{1}{K}\left(H\frac{d\rho_1}{ds}\right)\frac{d}{ds}\left(H\frac{d\rho_1}{ds}\right)+\rho_1\frac{d\rho_1}{ds}+K\frac{dK}{ds},$$

the last member of which is seen to be zero if we differentiate (36). Consequently, the center of the osculating hypersphere is fixed; and, since by hypothesis its radius is constant, it is the same at all points of the curve, and the curve itself is hyperspherical. Hence, the necessary and sufficient condition that a curve of triple curvature be hyperspherical is that the radius of its osculating hypersphere be constant. The equation expressing this condition is (36), but this, on account of (34), may be replaced by

$$\frac{H}{\rho}\frac{d\rho_1}{ds}+\frac{dK}{ds}=0,$$

which is characteristic of hyperspherical curves.

Involutes of curves of triple curvature.

20. If a curve L be such that each of its tangents is normal to another curve L_2, we say that L_2 is the involute of L, and that L is the evolute of L_2.

At any point of L, construct the usual tetrahedroid. According to the above definition, the involute L_2 is traced out by such a point on the axis of x of T that its locus is always perpendicular to the x-axis. Let 1, 0, 0, 0 be the coordinates of this point referred to the moving axes. By using (15), we find the projections of its velocity to be

$$\left.\begin{aligned} V_x &= 1+\frac{dl}{ds}, \\ V_y &= \frac{l}{\rho_1}, \\ V_z &= V_w = 0. \end{aligned}\right\} \qquad (37)$$

Expressing that the velocity is always perpendicular to the x-axis, we have

$$dl+ds=0 \text{ or } l+s=\text{const.}, \qquad (38)$$

a property which holds for curves of double curvature.

Equations (37) also show that the direction of the velocity of the point describing the involute is that of the principal normal of the evolute, and its magnitude is l/ρ_1.

Comparing this with what precedes, we see that the tangent to the involute, the principal normal to the evolute and the tangent to the spherical indicatrix of the evolutes, are parallel lines. Also, the velocity of the point describing the involute is l times that of the corresponding point on the spherical indicatrix mentioned.

If we define the rectifying hyperplane at any point of a curve of triple curvature as the hyperplane perpendicular to the principal normal at that point, we can say that the rectifying hyperplane of the evolute is the normal hyperplane of the involute.

Evolutes of a curve L.

21. Consider the tetrahedroid T attached in the usual manner to the curve L at a point $M(x, y, z, w)$. The evolute of L_2 must be traced out by such a point M_1 in the normal hyperplane of L that the tangent to its locus must always pass through M. Let the coordinates of M_1 with respect to the moving axes be 0, y_1, z_1, w_1; then, using (15) we find the projections of its velocity to be

$$\left.\begin{aligned} V_x &= 1 \quad - p_{12}y_1, \\ V_y &= \frac{dy_1}{ds} - p_{23}z_1, \\ V_z &= \frac{dz_1}{ds} + p_{23}y_1 - p_{34}w_1, \\ V_w &= \frac{dw_1}{ds} + p_{34}z_1. \end{aligned}\right\} \tag{37}$$

The first of these gives immediately

$$y_1 = \rho_1;$$

that is, the point M_1 is on the polar line and consequently *all the evolutes of a curve of triple curvature lie on the surface of regression of the polar hypersurface.*

If the tangent to the locus of M_1 is to pass through M we must have

$$\frac{\frac{dy_1}{ds} - p_{23}z_1}{y_1} = \frac{\frac{dz_1}{ds} + p_{23}y_1 - p_{34}w_1}{z_1} = \frac{\frac{dw_1}{ds} + p_{34}z_1}{w_1},$$

$$\left.\begin{aligned} y_1dz_1 - z_1dy_1 + p_{23}(y_1^2 + z_1^2)\,ds - p_{34}w_1y_1ds = 0, \\ z_1dw_1 - w_1dz_1 + p_{34}(w_1^2 + z_1^2)\,ds - p_{23}w_1y_1ds = 0. \end{aligned}\right\} \quad (40)$$

Projecting the first of these on the plane of yz gives

$$d.\arctan\frac{z_1}{y_1} = \frac{y_1dz_1 - z_1dy_1}{y_1^2 + z_1^2} = -p_{23}ds,$$

which may be written

$$d.\arctan\frac{z_1}{y_1} = \frac{ds}{H}.$$

On integrating, we find

$$\phi_1 = \arctan\frac{z_1}{y_1} = \int\frac{ds}{H}, \quad (41)$$

and, therefore,

$$z_1 = y_1\tan\phi_1 = \rho_1\tan\phi_1.$$

Similarly, from the second equation (40) we find

$$\phi_2 = \arctan\frac{w_1}{z_1} = \int\frac{ds}{\rho_3}, \quad (42)$$

and hence,

$$w_1 = z_1\tan\phi_2 = \rho_1\tan\phi_1\tan\phi_2.$$

If we denote the coordinates of M_1 with respect to the fixed axes by X_1, Y_1, Z_1, W_1, we have

$$X_1 = x + \beta_1\rho_1 + \gamma_1\rho_1\tan\phi_1 + \delta_1\rho_1\tan\phi_1\tan\phi_2,$$
$$Y_1 = y + \beta_2\rho_1 + \gamma_2\rho_1\tan\phi_1 + \delta_2\rho_1\tan\phi_1\tan\phi_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which, when $\rho_3 = \infty$, readily reduce to the corresponding formulæ for ordinary space.

Equations (41) and (42) show that ϕ_1 and ϕ_2 contain additively an arbitrary constant. If, then, we are given a family of normals touching a known evolute, and if we rotate each normal about the tangent at its point of incidence in such a way that ϕ_1 and ϕ_2 are increased by fixed amounts, we get a second family of normals touching a second evolute. Any curve of triple curvature has, therefore ∞^2 evolutes.

These same formulæ show that the principal normals of a curve of triple curvature cannot have an envelope; for, in order that ϕ_1 and ϕ_2 be zero, we must have

$$\rho_3 = \rho_2 = \infty,$$

and the curve would reduce to a plane curve.

WILLIAMS COLLEGE, WILLIAMSTOWN, MASS.

Primary Prime Functions in Several Variables and a Generalization of an Important Theorem of Dedekind.

By Harris Hancock.

In the usual theory of numbers, all integers are divided into two classes: the prime numbers and the composite numbers. By a finite number of trials one can determine whether an integer is prime or composite. In the theory of algebraic functions we have to consider whether a polynomial in x with integral coefficients is or is not decomposable into factors, and this determination is also effected by a finite number of trials. Methods of limiting the number of trials have been given by Kronecker,* Runge,† and Mandl,‡ who at the same time show how to find the factors when the polynomial is reducible into factors.

The decomposition of polynomials in two or more variables with integral coefficients into their irreducible factors, has been treated by Meyer‖ and the author.§

The next problem which we have to consider is the decomposition of polynomials in one variable with integral coefficients into their irreducible factors, when the coefficients are taken with respect to a modulus which is a prime integer. For example, the polynomial

$$f(x) = a_0x^r + a_1x^{r-1} + \ldots . + a_r,$$

where the integers $a_0, a_1, \ldots . a_r$ are taken with respect to the modulus p, is decomposable into factors when it is possible to find three integral functions in x with integral coefficients $f_1(x)$, $f_2(x)$ and $\phi_1(x)$ such that

$$f(x) = f_1(x) f_2(x) + p\phi_1(x),$$

or

$$f(x) \equiv f_1(x) f_2(x) \pmod{p}.$$

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* Kronecker, "Grundzüge," etc., §4, p. 11.

† Runge, Crelle, Bd. XCIX, p. 89.

‡ Mandl, Crelle, Bd. CXIII, p. 252.

‖ Meyer, Math. Ann., Bd. XXX, p. 30.

§ Hancock, Ann. de l'École Norm. Sup., t. XVII, p. 89.

In this case we say that $f(x)$ is *decomposable into factors* (mod. p); otherwise we say that $f(x)$ is *irreducible* (mod. p).

In Crelle's Journal, Bd. CXXII, p. 269, the author has indicated by a simple example the methods which are to be employed in the general case.

In the same way we may consider a polynomial $F(x, y)$ in two variables with integral coefficients, where the polynomial, when expanded, has the form

$$F(x, y) = a_0(x)\, y^\rho + a_1(x)\, y^{\rho-1} + \ldots\ldots + a_{\rho-1}(x)\, y + a_\rho(x).$$

We suppose that the coefficients $a_0(x), a_1(x), \ldots\ldots a_\rho(x)$ are taken with respect to an integral function in x with integral coefficients, $g_1(x)$, say, and that the integral coefficients in $g_1(x)$ have been reduced with respect to the prime integer p. We further assume that $g_1(x)$ is irreducible (mod. p).

We say that $F(x, y)$ is *reducible into factors with respect to the prime integer p and the irreducible (mod. p) function* $g_1(x)$ when two factors $F_1(x, y)$ and $F_2(x, y)$ can be found so that

$$F(x, y) = F_1(x, y)\, F_2(x, y) + p\phi_1(x, y) + g_1(x)\, \psi_1(x, y),$$

or
$$F(x, y) \equiv F_1(x, y)\, F_2(x, y) \text{ [modd. } p, g_1(x)],$$

where $\phi_2(x, y)$ and $\psi_2(x, y)$ are two polynomials integral in x, y, with integral coefficients; otherwise $F(x, y)$ is *irreducible* with respect to the modular system $[p, g_1(x)]$.

In the same manner, if $g_2(x, y)$ is an integral function in x, y with integral coefficients, and is irreducible with respect to the modular system $[p, g_1(x)]$, we say that *a function $G(x, y, z)$ integral in x, y, z with integral coefficients is decomposable into factors with respect to the three moduli,* p, $g_1(x)$, $g_2(x, y)$, when we can find functions $G_1(x, y, z)$, $G_2(x, y, z)$, $\phi_3(x, y, z)$, $\psi_3(x, y, z)$, $\chi_3(x, y, z)$ integral in these variables with integral coefficients such that

$$G(x, y, z) = G_1(x, y, z)\, G_2(x, y, z) + p\phi_3(x, y, z) + g_1(x)\, \psi_3(x, y, z) + g_2(x, y)\, \chi_3(x, y, z),$$

or

$$G(x, y, z) \equiv G_1(x, y, z)\, G_2(x, y, z) \text{[modd. } p, g_1(x), g_2(x, y)].$$

When such functions cannot be found, the function $G(x, y, z)$ is *irreducible* with respect to the modular system $[p, g_1(x), g_2(x, y)]$.

Such conceptions may be extended as far as we choose.

In the first case considered above we have to consider polynomials in one variable taken with respect to the prime integer p as a modulus; in the second case we consider polynomials in two variables with respect to the *prime* modular system $[p, g_1(x)]$, cf. Kronecker (Werke III^1, p. 158); in the third case polynomials in three variables with respect to the *prime* modular system $[p, g_1(x), g_2(x, y)]$, etc.

Let us take the polynomial

$$f(x) = a_0 x^\tau + a_1 x^{\tau-1} + \ldots\ldots + a_\tau,$$

when $a_i\,(i = 0, 1, \ldots\ldots \tau)$ are integers that are not divisible by p. Then it is always possible to find two other integers, positive or negative, $\overline{a_0}$ and $\overline{p}$ such that

$$a_0\overline{a_0} + p\overline{p} = 1.$$

Hence instead of considering functions like $f(x)$ with respect to the modulus p, we may consider the function $a_0 f(x) = f_0(x)$, say, where

$$f_0(x) = 1.\ x^\tau + b_1 x^{\tau-1} + b_2 x^{\tau-2} + \ldots\ldots + b_\tau \text{ (mod. } p),$$

and the integers $b_1, b_2, \ldots\ldots b_\tau$ have all been reduced (mod. p) and consequently are to be found among the numbers 0, 1, 2, $\ldots\ldots p-1$. For it is seen that if we multiply the function $f_0(x)$ in turn by the complete system of incongruent (mod. p) numbers 1, 2, $\ldots\ldots p-1$, we have the same $p-1$ functions (mod. p) as we have when $f(x)$ is multiplied by these numbers.

Next write

$$F(x, y) = a_0(x)\,y^\rho + a_1(x)\,y^{\rho-1} + \ldots\ldots + a_\rho(x),$$

where $a_i(x)\,[i = 0, \ldots\ldots \rho]$ are integral functions in x with integral coefficients and consider this function with respect to the modular system $[p, g_1(x)]$.

We assume that none of the function $a_0(x), a_1(x) \ldots\ldots a_\rho(x)$ is divisible by the modular system $[p, g_1(x)]$, that is, we cannot find two functions $\alpha(x)$ and $\beta(x)$ integral in x with integral coefficients so that

$$a_0(x) = p\,\alpha(x) + g_1(x)\,\beta(x).$$

This assumption granted, since $g_1(x)$ can have no factor in common with $a_0(x)$, it is always possible to find two functions $\overline{a_0}(x)$ and $\overline{g_1}(x)$ integral in x with integral coefficients, such that

$$a_0(x)\,\overline{a_0}(x) + g_1(x)\,\overline{g_1}(x) = m,$$

where m is an integer.

We can further find two integers $\overline{m}$ and π so that

$$m\overline{m} = 1 + p\pi.$$

Hence
$$\overline{m}\, a_0(x)\,\overline{a}_0(x) + \overline{m}\, g_1(x)\,\overline{g}_1(x) = 1 + p\pi,$$

and finally
$$\overline{m}\,\overline{a}_0(x)\, a_0(x) \equiv 1 \text{ [mod. } p, g_1(x)].$$

Consequently writing $\overline{m}\,\overline{a}_0(x)\, F(x, y) = F_0(x, y)$, we have

$$F_0(x, y) = 1\, y^\rho + b_1(x)\, y^{\rho-1} + \ldots. + b_\rho(x) \text{ [modd. } p, g_1(x)].$$

By means of the function $g_1(x)$ the degrees of the functions $b_1(x), \ldots. b_\rho(x)$ have all been reduced so as to be less than the degree of $g_1(x)$ and owing to the presence of p, all the numerical coefficients that appear are less than this prime integer.

Suppose next that

$$g_1(x) = 1.\ x^{n_1} + \gamma_1 x^{n_1-1} + \gamma_2 x^{n_1-2} + \ldots. + \gamma_{n_1-1} x + \gamma_{n_1},$$

where the integers $\gamma_1, \gamma_2, \ldots. \gamma_{n_1-1}, \gamma_{n_1}$ are found among the integers 0, 1, 2, $\ldots. p-1$.

By giving in turn these p values to each of the integers $\gamma_1, \gamma_2, \ldots. \gamma_{n_1}$ we derive $p^{n_1} = p_1$, say, functions which form the complete system of incongruent residues with respect to the modular system $[p, g_1(x)]$.

If we multiply the function $F_0(x, y)$ in turn by the $\varkappa = p^{n_1} - 1$ functions other than $g_1(x)$ we derive the same system of functions with respect to the modular system $[p, g_1(x)]$ as is had when we multiply the original function $F(x, y)$ by this same system of functions (cf. Crelle's Journal, Bd. CXXII, p. 271).

Proceeding in the same manner let $g_2(x, y)$ be an irreducible function with respect to the moduli p, $g_1(x)$, and with respect to the prime modular system $[p, g_1(x), g_2(x, y)]$ consider the function

$$G(x, y, z) = a_0(x, y)\, z^\sigma + a_1(x, y)\, z^{\sigma-1} + \ldots. + a_\sigma(x, y),$$

where $a_0(x, y), a_1(x, y), \ldots. a_\sigma(x, y)$ are integral functions in x, y with integral coefficients and are not divisible by the modular system $[p, g_1(x), g_2(x, y)]$.

We can always determine a multiplier $m_1\, \phi_1(x)\, \overline{a_0}(x, y)$, where m_1 is an integer, $\phi_1(x)$ is an integral function in x with integral coefficients and $\overline{a_0}(x, y)$ is an integral function in x, y with integral coefficients such that

$$m_1 \phi_1(x)\, \overline{a_0}(x, y)\, a_0(x, y) \equiv 1 \text{ [modd. } p, g_1(x), g_2(x, y)].$$

Consequently if we write $m\phi_1(x)\,\overline{a_0}(x, y)\,G(x, y, z) = G_0(x, y, z)$, it is seen that $G_0(x, y, z)$ has the form

$$G_0(x, y, z) \equiv z^\sigma + b_1(x, y)\,z^{\sigma-1} + b_2(x, y)\,z^{\sigma-2} + \ldots . + b_\sigma(x, y)\ [\text{modd. } p, g_1(x), g_2(x, y)],$$

where $b_1(x, y), b_2(x, y), \ldots . b_\sigma(x, y)$ have been reduced with respect to the modular system $[p, g_1(x), g_2 x, y)]$.

Suppose next that with respect to the modular system $[p, g_1(x)]$ that the function $g_2(x, y)$ has the form

$$g_2(x, y) = 1y^{n_2} + \delta_1(x)\,y^{n_2-1} + \delta_2(x)\,y^{n_2-2} + \ldots . + \delta_{n_2}(x),$$

where $\delta_1(x), \delta_2(x), \ldots . \delta_{n_2}(x)$ may be any of the p^{n_1} functions that constitute the complete system of incongruent residues with respect to the modular system $[p, g_1(x)]$.

There are consequently $[p^{n_1}]^{n_2} = p^{n_1 n_2} = p_2$ (say) such functions of the form $g_2(x, y)$ including, of course, the function $g_2(x, y)$ itself, and these p_2 functions constitute with respect to the modular system $[p, g_1(x), g_2(x, y)]$ a complete system of incongruent residues. It is also seen that the function $G(x, y, z)$ when multiplied in turn by the $p_2 - 1$ functions other than $g_2(x, y)$ produces the same system of functions with respect to the modular system $[p, g_1(x), g_2(x, y)]$ as those derived by the multiplication of $G_0(x, y, z)$ by the same system of functions.

Continuing this process let us form modular systems as follows: let p be a prime integer; then let $g_1(x_1)$ be an irreducible (mod. p) function of the n_1 degree in x_1 (all constant coefficients are once for all assumed integral); let $g_2(x_1, x_2)$ be irreducible [modd. p, $g_1(x_1)$] and of the n_2 degree in x_2 and form the modular system $[p, g_1(x_1), g_2(x_1, x_2)]$; let $g_3(x_1, x_2, x_3)$ be irreducible [modd. p, $g_1(x_1)$, $g_2(x_1, x_2)$] and of the n_3 degree in x_3, etc.

We may thus form the modular system

$$[M] = [p, g_1(x_1), g_2(x_1, x_2), \ldots . g_m(x_1, x_2, \ldots . x_m)].$$

With respect to this modular system there are a system of $p^{n_1 n_2 \ldots . n_m} = p_m$ incongruent residues which we may denote by $r_1(x_1, x_2, \ldots . x_m), r_2(x_1, x_2, \ldots . x_m), \ldots . r_{p_m}(x_1, x_2, \ldots . x_m)$, of which one is the function $g_m(x_1, x_2, \ldots . x_m)$ itself.

With respect to these residues and the above modular system, there exist the same relations for the integral functions in these m variables as those

expressed by the theorems of Fermat, Wilson, etc., in the theory of rational whole numbers, where the integers are taken with respect to a prime integer as modulus. Indeed, all the theorems given in §§26–31 of Dirichlet's "Zahlentheorie" (Dedekind, 4th edition), regarding congruences, primitive roots, etc., have their exact analogons here.

We come next to the consideration of functions $f(x_1, x_2, \ldots, x_m, x)$ taken with respect to the modular system $[M]$.

As above, it is seen that we may multiply this function by another function $\overline{f}(x_1, x_2, \ldots, x_m)$ say, so that when expanded in descending powers of x, the function $\overline{f}(x_1, x_2, \ldots x_m)\, f(x_1, x_2, \ldots x_m, x) = F(x_1, x_2, \ldots x_m, x)$ has the form

$$F(x_1, x_2, \ldots x_m, x) = 1 \cdot x^{\pi} + a_1(x_1, x_2, \ldots x_m)\, x^{\pi-1} + \ldots + a_{\pi}(x_1, x_2, \ldots x_m).$$

Such functions may be called *primary* functions in the $m+1$ variables $x_1, x_2, \ldots x_m, x$.

Regarding primary functions in one variable, see Galois (Journal de Math., t. XI, p. 398) Dedekind (XI Supplement to Dirichlet's "Zahlentheorie," p. 571 and other references of his works there cited; also, the numerous applications in C. Jordan's "Traité des Substitutions," Paris, 1870).

The first question which presents itself is: *to determine the number of primary functions in which the variable x occurs to the π^{th} degree, which are irreducible with respect to the modular system* $[M]$. Such functions are called *primary prime* functions.

The question may be expressed as follows: *to determine the number of primary prime functions that exist with respect to a given prime modular system, when the variable that does not appear in the modular system is of a given degree π, say.*

When $\pi = 1$, the function in question is

$$1) \quad x + a(x_1, x_2, \ldots x_m),$$

where $a(x_1, x_2, \ldots x_m)$ is an integral function in the variables and has been reduced by means of the modular system $[M]$.

There are consequently $p^{n_1 \cdot n_2 \cdots n_m} = p_m$ such functions, viz., the functions that are had when for $a(x_1, x_2, \ldots x_m)$ is replaced each of the p_m functions $r_1(x_1, x_2, \ldots x_m)$, $r_2(x_1, x_2, \ldots x_m)$, $\ldots$ $r_{p_m}(x_1, x_2, \ldots x_m)$.

If we denote by ω_1 the number of primary prime functions, where $\pi = 1$, we have*

$$\omega_1 = p_m.$$

When $\pi = 2$, we have the primary functions of the second degree in x, the form being

$$2) \quad x^2 + a(x_1, x_2, \ldots . x_m)\, x + b(x_1, x_2, \ldots . x_m),$$

where, in the places of $a(x_1, x_2, \ldots . x_m)$ and $b(x_1, x_2, \ldots . x_m)$, there may occur any of the p_m functions $r_1(x_1, x_2, \ldots . x_m), \ldots .$

There are consequently p_m^2 functions of the form 2), that is, primary functions of the second degree in x, where the coefficients have been reduced with respect to the modular system $[M]$.

To determine which of these functions are prime, we have only to subtract from p_m^2 the number of primary functions which arise from the multiplication of two primary functions of the first degree. This number is, since repetitions are allowable, $\frac{p_m(p_m+1)}{2}$. Hence, if ω_2 denotes the number of primary prime functions of the second degree in x whose coefficients have been reduced with respect to the modular system $[M]$, we have

$$\begin{aligned} \omega_2 &= p_m^2 - \frac{p_m(p_m+1)}{2} \\ &= \tfrac{1}{2} p_m(p_m - 1). \end{aligned}$$

A primary function of the third degree in x has the form

$$x^3 + a(x_1, x_2, \ldots . x_m)\, x^2 + b(x_1, x_2, \ldots . x_m)\, x + c(x_1, x_2, \ldots . x_m),$$

and we have all primary functions of the third degree in x whose coefficients have been reduced with respect to the modular system $[M]$ when, in the places of the functions $a(x_1, x_2, \ldots . x_m)$, $b(x_1, x_2, \ldots . x_m)$ and $c(x_1, x_2, \ldots . x_m)$, we substitute any of the p_m functions $r_1(x_1, x_2, \ldots . x_m), \ldots .$

We thus have p_m^3 such primary functions. These functions are either prime functions or products of a prime function of the first degree and one of the second degree or products of three primary prime functions of the first degree.

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* The present method is similar to that employed in the lectures at Berlin in 1894 on "Zahlentheorie," 2nd Part, by Professor George Frobenius for functions in one variable. I take the opportunity here of thanking this eminent mathematician for his courtesy in allowing me to make use of these lectures.

The number of the products of a primary prime function of the first degree with a primary prime function of the second degree is $\omega_1\omega_2$; and the number of the products of any three primary prime functions of the first degree is equal to the number of combinations of p_m elements taken three at a time admitting repetitions, that is, equal to $\frac{p_m(p_m+1)(p_m+2)}{1.2.3}$.

Hence, denoting the number of primary prime functions of the third degree in x whose coefficients have been reduced with respect to the modular system $[M]$ by ω_3, we have

$$p_m^3 = \omega_3 + \omega_1\omega_2 + \frac{p_3(p_m+1)(p_m+2)}{1.2.3},$$

from which it follows that

$$\omega_3 = p_m^3 - \frac{p_m \cdot p_m(p_m-1)}{2} - \frac{p_m(p_m+1)(p_m+2)}{6}$$
$$= \frac{p_m^3 - p_m}{3}.$$

We go now to the general determination of the number of the primary prime functions of the degree h in the variable x whose coefficients have been reduced with respect to the modular system $[M]$.

The number of primary functions of the degree h is p_m^h, since each of the h coefficients of the primary functions may be replaced by any of the functions

$$r_1(x_1, x_2, \ldots x_m),\ r_2(x_1, x, \ldots x_m),\ \ldots\ r_{p_m}(x_1, x_2, \ldots x_m).$$

Such a primary function of degree h can be either a primary prime function of the degree h or a product of a primary prime functions of the first degree, b primary prime functions of the second degree, c primary prime functions of the third degree, etc., where $a, b, c, \ldots$ are arbitrary integers ($\geqslant 0$) which are subjected to the only condition that

$$a + 2b + 3c + \ldots = h.$$

Now in how many different ways is such a formation of a product of a primary prime functions of the first degree, b primary prime functions of the second degree, etc., possible?

There are ω_1 primary prime functions of the first degree; these may be combined taken together a at a time and allowing repetitions in

$$\frac{\omega_1(\omega_1+1)(\omega_1+2)\ldots\ldots(\omega_1+a-1)}{1.2.3.\ldots\ldots a},$$

different ways. Similarly the ω_2 primary prime functions of the second degree may, when b of them are taken together and allowing repetitions, be combined in

$$\frac{\omega_2(\omega_2+1)(\omega_2+2)\ldots\ldots(\omega_2+b-1)}{1.2.3.\ldots\ldots b}$$

ways, etc.

Hence a product of a primary prime functions of the first degree, b primary prime functions of the second degree, etc., can be formed in

$$\frac{\omega_1(\omega_1+1)\ldots\ldots(\omega_1+a-1)}{a!}\cdot\frac{\omega_1(\omega_1+1)\ldots\ldots(\omega_1+b-1)}{b!}\ldots\ldots$$

ways.

We define the symbol $\binom{x}{n}$ by the well known relation,

$$\binom{x}{n}=\frac{x(x-1)(x-2)\ldots\ldots(x-n+1)}{n!},$$

where x is quite arbitrary, while n is a positive integer.

By means of this symbol we may write the above product in the form

$$(-1)^a\binom{-\omega_1}{a}(-1)^b\binom{-\omega_2}{b}\ldots\ldots$$

If we take the summation of this product over all systems of numbers $a, b, c, \ldots\ldots$ which satisfy the relation

$$a+2b+3c+\ldots\ldots=h,$$

we have the number of all primary functions of degree h, i. e.

$$p_m^h=\sum\left\{(-1)^a\binom{-\omega_1}{a}(-1)^b\binom{-\omega_2}{b}\ldots\ldots\right\},$$

where the summation is to be taken over all systems of numbers $a, b, c, \ldots\ldots$ which satisfy the condition

$$a+2b+3c+\ldots\ldots=h.$$

Let x be a variable; we then have

$$x^a\, x^{2b}\, x^{3c} \ldots . = x^h,$$

and if we multiply the above expression by x^h, it follows that

$$(p_m x)^h = \sum \left\{ (-1)^a \binom{-\omega_1}{a} x^a (-1)^b \binom{-\omega_2}{b} x^{2b} \ldots . \right\},$$

where the summation is to be taken as before.
This formula is true for all values of

$$h = 0,\ +1,\ +2, \ldots . \textit{ ad infinitum.}$$

If y is an arbitrary variable, whose absolute value is < 1, then is

$$\frac{1}{1-y} = \sum_{h=0}^{h=\infty} y^h.$$

Accordingly, if we assume that $p_m|x| < 1$, then is

$$\frac{1}{1-p_m x} = \sum_{h=0}^{h=\infty} p_m^h x^h,$$

and, consequently,

$$\frac{1}{1-p_m x} = \sum_{h=0}^{h=\infty} \Big[\sum_{a,\, b,\, c,\, \ldots .} \left\{ (-1)^a \binom{-\omega_1}{a} x^a (-1)^b \binom{-\omega_2}{b} x^{2b} \ldots . \right\} \Big],$$
$$(a + 2b + 3c + \ldots . = h),$$

or $$\frac{1}{1-p_m x} = \sum_{(a)} \left\{ (-1)^a \binom{-\omega_1}{a} x^a \right\} . \sum_{(b)} \left\{ (-1)^b \binom{-\omega_2}{b} x^{2b} \right\} \ldots . ,$$

since, in this manner for the above double summation, we may write a product of an infinite number of single summations owing to the convergence of the series. Further, since, in the double summation, h is taken from 0 to ∞, it follows also in the single summations that we must also cause the summations to be made from 0 to ∞. From the binomial theorem it follows that

$$(1-x)^{-\omega_1} = \sum_{a=0}^{\infty} \left\{ (-1)^a \binom{-\omega_1}{a} x^a \right\},$$

$$(1-x^2)^{-\omega_2} = \sum_{b=0}^{\infty} \left\{ (-1)^b \binom{-\omega_2}{b} x^{2b} \right\},$$

$$(1-x^3)^{-\omega_3} = \sum_{c=0}^{\infty} \left\{ (-1)^c \binom{-\omega_3}{c} x^{3c} \right\},$$

etc.

Consequently, we have

$$1 - p_m x = (1-x)^{\omega_1}(1-x^2)^{\omega_2}(1-x^3)^{\omega_3}\ldots.$$
$$= \prod_{d=1}^{\infty}\left\{(1-x^d)^{\omega_d}\right\}.$$

From this equation we may easily determine the quantities $\omega_1, \omega_2, \omega_3, \ldots$ *ad infinitum, by equating like powers of* x.

We further have from this relation

$$\log(1-p_m x) = \log\prod_{d=1}^{\infty}\left\{(1-x^d)^{\omega_d}\right\}$$
$$= \sum_{d=1}^{\infty}\left\{\log(1-x^d)^{\omega_d}\right\}$$
$$= \sum_{d=1}^{\infty}\left\{\omega_d \log(1-x^d)\right\}.$$

If we differentiate this equation with respect to x, we have

$$\frac{p_m}{1-p_m x} = \sum_{d=1}^{\infty}\left\{d\omega_d \frac{x^{d-1}}{1-x^d}\right\};$$

and, multiplying by x, it follows that

$$\frac{p_m x}{1-p_m x} = \sum_{d=1}^{\infty}\left\{d\omega_d \frac{x^d}{1-x^d}\right\}.$$

But, owing to the identical relation,

$$\frac{x^d}{1-x^d} = \sum_{n=1}^{\infty} x^{dn},$$

the above equation becomes

$$\frac{p_m x}{1-p_m x} = \sum_{d=1}^{\infty}\left\{d\omega_d \sum_{n=1}^{\infty} x^{dn}\right\}$$
$$= \sum_{n=1}^{\infty}\left\{\sum_{d=1}^{\infty} d\omega_d x^{dn}\right\}.$$

On the other hand we have

$$\frac{p_m x}{1-p_m x} = \sum_{h=1}^{\infty}(p_m^h x^h),$$

and consequently it follows that

$$\sum_{n=1}^{\infty}\left\{\sum_{d=1}^{\infty}(d\omega_d\, x^{dn})\right\} = \sum_{h=1}^{\infty}(p_m^h\, x^h).$$

Since this equation must exist identically, we have by equating like powers of x on either side, the relation

$$p_m^h = \sum_{(d)} d\omega_d\ ,$$

where, in the summation d is to be taken over all values for which $h = d_n$, that is, d is taken over all divisors of h, including h.

An immediate application of the above results is to be found in an extension of a theorem due to Dedekind, a theorem which in the further development of the theory of algebraic numbers will hold the same important position as do the theorems of Fermat and Wilson in the ordinary theory of rational numbers.

Consider the algebraic numbers that belong to a fixed realm of rationality* (Körper), which denote by Ω and let v be the totality (Dedekind, p. 537) of all the algebraic integers of the realm of rationality Ω.

This modul v (see Dedekind, p. 537) has the same relations with respect to the realm Ω as unity has in the realm of rational numbers, for example

$$v.\, v = v,$$

$$\frac{v}{v} = v,$$

and every algebraic integer in Ω is divisible by the modul v.

Further let p be a rational prime integer that is divisible by the prime ideal $\mathfrak{p}$ [in v] (Dedekind, p. 560).
Then the norm of $\mathfrak{p}$ is

$$N(\mathfrak{p}) = p^f,$$

where f is a rational integer such that

$$0 < f \leq n,$$

n being the degree of the realm Ω (Dedekind, p. 565).

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* Dedekind, XI Supplement to Dirichlet's "Zahlentheorie," fourth edition, p. 452.
We shall in the following refer to this supplement by using merely the name of the author.

The number f is called the degree of the ideal $\mathfrak{p}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two arbitrary moduls, then the number of classes into which the algebraic integers that are contained in $\mathfrak{a}$ may be distributed with respect to $\mathfrak{b}$ is represented by the symbol*

$$(\mathfrak{a}, \mathfrak{b}) = m, \text{ say.}$$

If we choose a definite integer out of each class, we have a complete *representative system* of the modul $\mathfrak{a}$ with respect to the modul $\mathfrak{b}$.

These m integers may be denoted by $\rho_1, \rho_2, \ldots \rho_m$. They have the following characteristics:

1) Each of the m numbers is divisible by $\mathfrak{a}$.

2) The difference of any two of these numbers is not divisible by $\mathfrak{b}$.

3) Every number that is divisible by $\mathfrak{a}$ is congruent to one of these m numbers (mod. $\mathfrak{b}$).

Dedekind (p. 564) shows that

$$(v, \mathfrak{p}) = N(\mathfrak{p}) = p^f.$$

Let $p^f = \varkappa$ and let $\rho_1, \rho_2, \ldots \rho_\varkappa$ be a complete representative system of the integers contained in the modul v with respect to the modul $\mathfrak{p}$. As unity is to be found among the integers in v, we take unity as the representative of the class to which it belongs and write one of the numbers $\rho_1, \rho_2, \ldots \rho_\varkappa$ equal to unity, say $\rho_1 = 1$.

Dedekind (p. 570) further shows, if ω is any algebraic integer in Ω, that

$$\omega^p \equiv \omega \pmod{\mathfrak{p}}.$$

The following theorem follows at once:

$$x^{p-1} - 1 \equiv \Pi(x - \omega) \pmod{\mathfrak{p}},$$

where x is a variable and the product is to be taken over a system of integers incongruent and relative prime to $\mathfrak{p}$.

Since this congruence exists identically for all values of x, we have, when $x = 0$, the analogon of Wilson's theorem in the theory of rational numbers:

$$\Pi\omega = -1 \pmod{\mathfrak{p}},$$

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* Dedekind, p. 509.

where the product is to be taken over a complete system of integers ω which are incongruent (mod. $\mathfrak{p}$) and are relative prime to $\mathfrak{p}$.

If α is an algebraic integer in Ω, and if

$$\alpha^{p^h} \equiv \alpha \ [\text{mod. } \mathfrak{p}],$$

where h is the smallest rational integer that satisfies this congruence, then h is called the *height* (Dedekind, p. 571) of the integer α with respect to the modulus $\mathfrak{p}$.

It is clear that $h \leq f$.

Consider the infinite series of numbers

$$\alpha^{p^0}, \alpha^{p^1}, \alpha^{p^2}, \alpha^{p^3}, \ldots .$$

with respect to the modulus $\mathfrak{p}$, and let h be the height of α. The first h of these numbers are incongruent (mod. $\mathfrak{p}$), and it is seen that after the first h of these numbers, the next h of them are repeated in the same sequence, etc.

The h numbers $\quad \alpha^{p^0}, \alpha^{p^1}, \alpha^{p^2}, \ldots . \alpha^{p^{h-1}}$

are called the *period* of the number α.

It may be proved that the congruence

$$x^{p^h} - x \equiv 0 \ (\text{mod. } \mathfrak{p})$$

has exactly p^h incongruent (mod. $\mathfrak{p}$) roots; and, indeed, this congruence is satisfied by all quantities α whose height is h or a divisor of h, the number of such quantities being p^h.

We therefore have

$$1) \quad x^{p^h} - x \equiv \Pi (x - \alpha) \ (\text{mod. } \mathfrak{p}),$$

where the product is to be taken over a system of p^h incongruent (mod. $\mathfrak{p}$) integers whose height is h or a divisor of h.

In this product there appears, if α is an integer whose height is h, the product

$$(x - \alpha)(x - \alpha^p)(x - \alpha^{p^2}) \ldots . (x - \alpha^{p^{h-1}}),$$

a product which we may replace by a congruent (mod. $\mathfrak{p}$) primary prime function of the h^{th} degree in x with rational integral coefficients.

In the same way, there enters in the product 1) a product of the form

$$(x-\beta)(x-\beta^{p})(x-\beta^{p^2}) \ldots\ldots (x-\beta^{p^{d-1}}),$$

if β is an integer whose height is d; this latter product may also be replaced by a congruent (mod. $\mathfrak{p}$) primary prime function of degree d with rational integral coefficients.

It is thus seen that

$$x^{p^h} - x \equiv \Pi P(x) \pmod{\mathfrak{p}},$$

where, since $p^h = \sum_{(d)} d\omega_d$, the product is to be taken over all the existing primary prime functions whose height is h or a divisor of h (Dedekind, p. 572).

The preceding investigations have been made for algebraic integers in the fixed realm of rationality Ω. We denote the realm containing all these integers by $[v]$.

Instead of considering algebraic integers we shall now consider integral functions of several variables $x_1, x_2, \ldots\ldots x_m$ whose coefficients are those algebraic integers. We say that such functions belong to the realm of integrity $[v, x_1, x_2, \ldots\ldots x_m]$, this realm being composed of all integral functions in these variables with constant coefficients that belong to the realm $[v]$.

Let us consider first the functions belonging to the realm $[v, x_1]$ and sup pose that

$$f(x_1) = \alpha_0 x_1^n + \alpha_1 x_1^{n-1} + \ldots\ldots + \alpha_n,$$

where $\alpha_0, \alpha_1, \ldots\ldots \alpha_n$ are algebraic integers in the realm $[v]$.

Taken with respect to the prime ideal $\mathfrak{p}$, the algebraic integers in v may be distributed, as shown above, into

$$(v, \mathfrak{p}) = p^f = \varkappa \text{ classes.}$$

Further since the norm of $\mathfrak{p}$, that is $N(\mathfrak{p}) = \varkappa$, it follows that the rational integers $N(\rho_1), N(\rho_2), \ldots\ldots N(\rho_\varkappa)$ are all less than $\varkappa$, the algebraic integers $\rho_1, \rho_2, \ldots\ldots \rho_\varkappa$ being as defined above.

In another paper* the author has shown that one of the representatives ρ_i, say, is such that

$$\rho_i \alpha_0 \equiv 1 \pmod{\mathfrak{p}},$$

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* Mémoire sur les systèmes modulaires de Kronecker (Ann. de l'École Norm. Sup., t. XVIII, Supplement I, p. 64).

and further with respect to the modulus $\mathfrak{p}$, we have the same system of functions when we multiply the function $f(x_1)$ in turn by the numbers $\rho_1, \rho_2, \ldots \rho_\kappa$ as we have when the function $\rho_i f(x_1)$ is multiplied by this same system of numbers. Hence instead of considering the original function $f(x_1)$ with respect to the modulus $(\mathfrak{p})$, we may consider the function $f_0(x_1) = \rho_i f(x_1)$ with respect to this modulus.

The function $f_0(x_1)$ has the form

$$f_0(x_1) = x_1^n + \overline{\alpha}_1 x_1^{n-1} + \overline{\alpha}_2 x_1^{n-2} + \ldots + \overline{\alpha}_n,$$

where $\overline{\alpha}_1, \overline{\alpha}_2, \ldots \overline{\alpha}_n$ are to be found among the integers $\rho_1, \rho_2, \ldots \rho_\kappa$.

It is clear that the number of functions having the form $f_0(x_1)$ is $\varkappa^n$.

This then is the number of incongruent functions of degree n in the realm $[v, x_1]$ with respect to the modulus $\mathfrak{p}$ and these functions together with the prime ideal $\mathfrak{p}$ constitute the modular system

$$[\mathfrak{p}, f_0(x_1)].$$

Next, suppose that $g_1(x_1)$ is an irreducible function with respect to the modulus $\mathfrak{p}$ and is of the form

$$g_1(x_1) = x_1^{n_1} + \beta_1 x_1^{n_1-1} + \beta_2 x_1^{n_1-2} + \ldots + \beta_{n_1},$$

where $\beta_1, \beta_2, \ldots \beta_{n_1}$ are to be found among the integers $\rho_1, \rho_2, \ldots \rho_\kappa$.

Let us form the prime modular system

$$[\mathfrak{p}, g_1(x_1)].$$

Then the number of incongruent (modd. $\mathfrak{p}, g_1(x_1)$) functions in the realm $[v, x_1]$ is $\varkappa^{n_1}$.

Let $\phi(x_1, x_2)$ be an integral function in x_1, x_2 with integral coefficients that belong to $[v]$, and when expanded in descending powers of x_2 suppose that the form of $\phi(x_1, x_2)$ is

$$\phi(x_1, x_2) = d_0(x_1) x_2^\gamma + d_1(x_1) x_2^{\gamma-1} + \ldots + d_\gamma(x_1).$$

With respect to the modular system $[\mathfrak{p}, g_1(x_1)]$ we may always find a function $\overline{d_0}(x_1)$ integral in x_1 with coefficients that belong to $[v]$ such that

$$\overline{d_0}(x_1) d_0(x_1) \equiv 1 \ [\text{modd. } \mathfrak{p}, g_1(x_1)].$$

Hence, with respect to this modular system, the function $\overline{d_0}(x_1)\,\phi(x_1, x_2) = \phi_0(x_1, x_2)$ is of the form

$$\phi_0(x_1, x_2) = x_2^\gamma + \overline{d_1}(x_1)\,x_2^{\gamma-1} + \overline{d_2}(x_1)\,x_2^{\gamma-2} + \ldots + d_\gamma(x_2),$$

where the functions $\overline{d_1}(x_1)$, $\overline{d_2}(x_1)$, $\ldots$ $\overline{d_\gamma}(x_2)$ belong to the realm $[v, x_1]$ and are of degrees at most $= n_1 - 1$ in x_1.

Consequently, the number of functions of this form is $\varkappa^{n_1\gamma}$, and these functions form a system of incongruent residues in the realm $[v, x_1, x_2]$ with respect to the modular system $[\mathfrak{p}, g_1(x_1), \phi_0(x_1, x_2)]$.

Next suppose that, taken with respect to the modular system $[\mathfrak{p}, g_1(x_1)]$, the function

$$g_2(x_1, x_2) = x_2^{n_2} + \beta_1(x_1)\,x^{n_2-1} + \beta_2(x_1)\,x^{n_2-2} + \ldots + \beta_{n_2}(x_1)$$

is irreducible, and form the prime modular system

$$[\mathfrak{p}, g_1(x_1), g_2(x_1, x_2)].$$

With respect to this modular system, the number of incongruent functions in the realm $[v, x_1, x_2]$ is $\varkappa^{n_1 \cdot n_2}$.

Continuing this process, let

$$g_3(x_1, x_2, x_3) = x_3^{n_3} + \gamma_1(x_1, x_2)\,x_3^{n_3-1} + \ldots + \gamma_{n_3}(x_1, x_2)$$

be an irreducible function with respect to the modular system $[\mathfrak{p}, g_1(x_1), g_2(x_1, x_2)]$ where the coefficients $\gamma_1(x_1, x_2)$, $\ldots$ $\gamma^{n_3}(x_1, x_2)$ belong to the realm $[v, x_1, x_2]$.

Then, with respect to the prime modular system

$$[\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), g_3(x_1, x_2, x_3)],$$

there are $\varkappa^{n_1 \cdot n_2 \cdot n_3}$ incongruent functions in the realm $[v, x_1, x_2, x_3]$.

By extending this process, we form the prime modular system

$$[M] \quad [\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), g_3(x_1, x_2, x_3), \ldots g_m(x_1, x_2, \ldots x_m)],$$

where the function $g_i(x_1, x_2, \ldots x_i)$ is irreducible with respect to the modular system $[p, g_1(x_1), g_2(x_1, x_2), \ldots g_{i-1}(x_1, x_2, \ldots x_{i-1})]$,

$$(i = 2, 3, \ldots m)$$

and is of degree n_i in x_i.

Then with respect to the modular system $[M]$ there are $\varkappa^{n_1 \cdot n_2 \cdots n_m}$ incongruent functions in the realm $[v, x_1, x_2, \ldots x_m]$.

For brevity write

$$\varkappa_i = \varkappa^{n_1 \cdot n_2 \cdots n_i} (i = 1, 2, \ldots m)$$

and let us denote the above residues by

$$r_1(x_1, x_2, \ldots x_m), r_2(x_1, x_2, \ldots x_m), \ldots r_{\kappa_m}(x_1, x_2, \ldots x_m),$$

of which one is the function $g_m(x_1, x_2, \ldots x_m)$.

Omitting for the moment this residue from the consideration, it is seen that if $\zeta(x_1, x_2, \ldots x_m)$ is any quantity belonging to the realm $[v, x_1, x_2, \ldots x_m]$ then is

$$[\zeta(x_1, x_2, \ldots x_m)]^{\kappa_m - 1} \equiv 1 \ [\text{modd. } \mathfrak{p}, g_1(x_1) \ldots g_m(x_1, x_2, \ldots x_m)].$$

THEOREM. *If r is any of the above residues $r_1, r_2, \ldots r_{\kappa_m}$, where we have used the functional signs instead of the functions themselves and if*

$$G(r) = A_0 r^\nu + A_1 r^{\nu-1} + \ldots + A_\nu \equiv 0 \ [\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

where $A_0, A_1, \ldots A_\nu$ are functions of the realm $[v, x_1, x_2, \ldots x_m]$, and if further $A_0 \not\equiv 0 \ [\text{modd. } \mathfrak{p}, g_1(x_1), \ldots g_m(x_1, x_2, \ldots x_m)]$, then the congruence 1) cannot have more than ν incongruent $[\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)]$ roots; if further

$$G(r) \equiv 0 \ [\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

has exactly ν incongruent roots $R_1, R_2, \ldots R_\nu$, say, with respect to this modular system, we have the identical congruence

$$G(r) \equiv A_0 \prod_{i=1}^{i=\nu} (r - R_i) \ (\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)].$$

Hence since

$$r^{\kappa_m} - r \equiv 0 \ [\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

has the $\varkappa_m$ incongruent $[\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)]$ roots $r_1, r_2, \ldots r_{\kappa_m}$ above, it is seen that we have the identical congruence:

$$r^{\kappa_m - 1} - 1 \equiv \prod_{\lambda} (r - r_\lambda) \text{ [modd. } \mathfrak{p}, g_1(x_1)\, g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)]$$

where the congruence is to be taken over the $\varkappa_m - 1$ residues $r_1, r_2, \ldots\ r_{\varkappa_m}$ not including $g_m(x_1, x_2, \ldots\ x_m)$.

Equating the coefficients of r on either side of this congruence, we have the very general *Wilson Theorem*

$$-1 \equiv \prod_{\lambda} r_\lambda \text{ [modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)],$$

where the product is to be taken as above.

Let $\xi(x_1, x_2, \ldots\ x_m)$ be any quantity of the realm $[v, x_1, x_2, \ldots\ x_m]$ so that therefore

$$[\xi(x_1, x_2, \ldots\ x_m)]^{\kappa_m} \equiv \xi(x_1, x_2, \ldots\ x_m) \text{ [modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)].$$

Hence if we write the functional sign for the function itself and give to $\varkappa_m$ its value $p^{f \cdot n_1 \cdot n_2 \ldots n_m} = p_m^f$, then is

$$[\xi]^{p_m^f} \equiv \xi \text{ [modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)].$$

Following the analogy of Dedekind we may call h the *height* of the function $\eta(x_1, x_2, \ldots\ x_m)$ with respect to the modular system $[\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)]$ when h is the smallest rational integer such that $[\eta]^{p_m^h} \equiv \eta$ [modd. $\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)$] and consequently $h \leq f$.

The functions

$$\eta^{p_m^0}, \eta^{p_m^1}, \eta^{p_m^2}, \ldots\ \eta^{p_m^{h-1}},$$

constitute the period of the function η with respect to the modular system $[\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)]$.

The product

$$(x - \eta^{p_m^0})(x - \eta^{p_m^1})(x - \eta^{p_m^2}) \ldots\ (x - \eta^{p_m^{h-1}}),$$

is congruent to

$$P(x, x_1, x_2, \ldots\ x_m) \text{ [modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots\ g_m(x_1, x_2, \ldots\ x_m)],$$

where $P(x, x_1, x_2, \ldots x_m)$ is a primary prime function of the h^{th} degree in x, of degrees not greater than $n_1 - 1$ in x_1, $n_2 - 1$ in x_2, $n_3 - 1$ in x_3, $\ldots$ $n_m - 1$ in x_m and whose constant coefficients are rational integers.

It is easy to prove that the congruence

$$x^{p_m^h} - x \equiv 0 \ [\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

has exactly p_m^h incongruent [modd. $\mathfrak{p}$, $g_1(x_1)$, $g_2(x_1, x_2)$, $\ldots$ $g_m(x_1, x_2, \ldots x_m)$] roots.

Hence we have

$$1) \qquad x^{p_m^h} - x \equiv \prod (x - \zeta) \ [\mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

where the product is to be taken over a system of $\mathfrak{p}_m^h$ incongruent [modd. $\mathfrak{p}$, $g_1(x_1)$, $g_2(x_1, x_2)$, $\ldots$ $g_m(x_1, x_2, \ldots x_m)$] quantities which have a height h or a height d, where d is a divisor of h.

In this product, if ζ is a function of the height h, there appears as a factor the product

$$(x - \zeta)(x - \zeta^{p_m})(x - \zeta^{p_m^2}) \ldots (x - \zeta^{p_m^{h-1}}),$$

a product which may be replaced by a primary prime function as seen above.

There appears also as a factor of the product 1), if d is a divisor of h, the product

$$(x - \zeta')(x - \zeta'^{p_m})(x - \zeta'^{p_m^2}) \ldots (x - \zeta'^{p_m^{d-1}}),$$

where ζ' is a function which has the height d.

This product may also be replaced by a primary prime function.

It is thus evident that we may replace the product 1) by a product of primary prime functions and we have in this manner

$$x^{p_m^h} - x \equiv \prod P(x, x_1, x_2, \ldots x_m)[\text{modd. } \mathfrak{p}, g_1(x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

where the product is to be taken over a certain number of primary prime functions.

Since every congruence between rational integers with respect to the modulus $\mathfrak{p}$, is also true when taken with respect to the modulus p, it follows, since all the coefficients in the primary prime functions are rational integers, that instead of the modular system

$$[\mathfrak{p}, g_1, (x_1), g_2(x_1, x_2), \ldots g_m(x_1, x_2, \ldots x_m)],$$

we may write the modular system

$$[p,\ G_1(x_1),\ G_2(x_1, x_2),\ \ldots.\ G_m(x_1, x_2,\ \ldots.\ x_m)],$$

where $G_i\ (x_1, x_2\ \ldots.\ x_i)$ are the resulting functions when in $g_i\ (x_1, x_2,\ \ldots.\ x_i)$ $[i = 1,\ 2,\ \ldots.\ m]$ we write instead of the algebraic integers $\rho_1,\ \rho_2,\ \ldots.\ \rho_\kappa$ which appear as coefficients the rational integers which belong to the same classes of which the integers $\rho_1,\ \rho_2,\ \ldots.\ \rho_\kappa$ have been taken as representatives; these rational integers are supposed reduced (mod. p).

We thus have the identical congruence

$$x^{p^h_m} - x \equiv \prod P(x, x_1, x_2, \ldots. x_m)[\text{modd.}\, p,\ G_1(x_1), G_2(x_1, x_2) \ldots. G_m(x_1 x_2 \ldots. x_m)],$$

where the product is taken over all primary prime functions of degree h or a divisor of h, the number of such functions being given by the formula (see p. 50):

$$p^h_m = \sum_{(d)} d\omega_d\,.$$

In this connection, attention is called to two papers by the author: 1°. *Mémoire sur les Systèmes Modulaires de Kronecker*, pp. 83 and 108, a paper which was presented as a doctor's thesis to the University of Paris and published *in extenso* in the Ann. de l'École Norm. Sup., t. XVIII. 2°. Some Remarks on Kronecker's Modular Systems, Compt. Rend. du Congrès des Mathématiciens. Paris. 1900.

I add a more general statement of a theorem first given in the above thesis, p. 42.

Let

$$1).\ [\mathfrak{p}, f_1(x), f_2(x),\ \ldots.\ f_n(x), h_1(x), h_2(x),\ \ldots.\ h_m(x)]$$

be a modular system in which $\mathfrak{p}$ is a prime ideal, and let the functions $f_1(x),\ f_2(x),\ \ldots. f_n(x), h_1(x), h_2(x),\ \ldots.\ h_m(x)$ belong to the realm $[v, x]$. Further let the functions $h_1(x), h_2(x),\ \ldots.\ h_m(x)$ be linearly dependent with respect to the modulus $\mathfrak{p}$ upon the functions $f_1(x), f_2(x),\ \ldots.\ f_n(x)$, i. e., let

$$h_i(x)\, k_i(x) = f_1(x)\, \phi_{1i}(x) + f_2(x)\, \phi_{2i}(x) + \ldots. + f_n(x)\, \phi_{ni}(x)\ [\text{mod. } \mathfrak{p}],$$
$$(i = 1,\ 2,\ \ldots.\ m),$$

where the functions $k_i(x),\ \phi_{1i}(x)\ \phi_{2i}(x),\ \ldots.\ \phi_{ni}(x)$ belong to the realm $[1, x]$.

Then, if we suppose that the functions $f_1(x), f_2(x), \ldots . f_n(x)$ are linearly independent with respect to the modulus $\mathfrak{p}$, it may be shown precisely in the same manner, as was given on p. 42 of the thesis, that the system 1) may be replaced by an equivalent system which contains only n independent elements with respect to the modulus $\mathfrak{p}$.

The theorem may be further generalized by considering functions of several variables taken with respect to the modular system $[M]$ of page 43.

UNIVERSITY OF CINCINNATI, *December* 21, 1900.

On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity.

By R. A. Roberts.

Part II.—*On Certain Plane Cubic Curves and their Angles of Intersection, with some Account of Conics Cutting Orthogonally.*

I propose to investigate here some methods of generating certain plane cubic curves in such a way that their angles of intersection assume a simple form.

Let us consider a system of conics passing through four fixed points. Let U, V be any two conics of the system, where

$$U = ax^2 + by^2 + 2hxy + 2gx + 2fy + c,$$
$$V = a'x^2 + b'y^2 + 2h'xy + 2g'x + 2f'y + c',$$

the axes being rectangular, then

$$U - \lambda V = 0 \tag{1}$$

is any conic of the system. Now, let us seek the locus of the points of contact of tangents of (1) drawn parallel to the axis of x. For these two points we must have

$$\frac{dU}{dx} - \lambda \frac{dV}{dx} = 0, \text{ or } L - \lambda L' = 0, \tag{2}$$

if L, M, N are the three first differential coefficients of U and L', M', N' the similar functions for V. Hence, for the locus we have

$$2\phi\, U = \frac{dV}{dx} - V\frac{dU}{dx} = 0. \tag{3}$$

Similarly, for the locus of the points of contact of the tangents parallel to the axis of y, we have

$$2\psi = U\frac{dV}{dy} - V\frac{dU}{dy} = 0, \tag{4}$$

and for the locus of the points of contact of tangents parallel to the line $lx + my = 0$, we have

$$l\psi - m\phi = 0. \tag{5}$$

These curves are evidently cubics passing through the four points $U=0$, $V=0$. Further, they pass through the intersection of the diagonals and opposite sides of the quadrangle formed by the four points; for, putting

$$2U = Lx + My + N,$$
$$2V = L'x + M'y + N',$$

as is identically the case, it is readily seen that both (3) and (4) are satisfied by

$$\frac{L'}{L} = \frac{M'}{M} = \frac{N'}{N}, \tag{6}$$

which gives the vertices of the common self-conjugate triangle of U, V, namely, the three points just mentioned.

Differentiating (3) and (4), we have

$$\frac{d\phi}{dx} = a'U - aV, \quad \frac{d\phi}{dy} = h'U - hV + 2J, \tag{7}$$

$$\frac{d\psi}{dx} = h'U - hV - 2J, \quad \frac{d\psi}{dy} = b'U - bV, \tag{8}$$

where

$$\frac{dV}{dx}\frac{dU}{dy} - \frac{dV}{dy}\frac{dU}{dx} = 4J. \tag{9}$$

Thus for the four points $U=V=0$, we have $\frac{d\phi}{dx} = 0$; that is, the tangents to the cubic at these points are parallel to the axis of x. Again, these tangents are parallel to an asymptote of the cubic, for, seeking the points at infinity on (3), we obtain

$$\left.\begin{aligned} &(a'x + h'y)(ax^2 + 2hxy + by^2) - (ax + hy)(a'x^2 + b'y^2 + 2h'xy) = 0, \\ \text{or} \quad &y\{(a'x + h'y)(hx + by) - (ax + hy)(h'x + b'y)\} = 0. \end{aligned}\right\} \tag{10}$$

The curve ϕ is then a general cubic described in a unique manner corresponding to a given direction; that is, the cubic passes through the four points so that its tangents at these points are parallel to an asymptote, there being but one cubic corresponding to a given direction, as we see by considering the fact that the cubic passes through the intersections of the diagonals and opposite sides. The latter points are always on the cubic, when the tangents at the four points pass through a point P on the curve (see Salmon's "Higher Plane Curves," §150), the tangents at the three points passing through the tangential of P. Hence, the cubics ϕ, ψ cut at right angles at the four points $U=V=0$, and the cubics $l\psi - m\phi = 0$, $l'\psi - m'\phi = 0$ cut, at the same four points, at an angle equal to that between the lines $lx + my = 0$, $l'x + m'y = 0$. It may be observed that the cubics ϕ, ψ or the cubics $l\psi - m\phi$, $l'\psi - m'\phi$ intersect at two other points beside the seven points already determined. These two points are the directions of the two other asymptotes, which are common to all the cubics, as we see from (10). These two directions are evidently those of the axes of the two parabolæ which can be described through the four points. Let us now seek the angles of intersection of the cubics ϕ, ψ at the vertices of the common self-conjugate triangle of U, V. For these points we have $J=0$ in (7) and (8), and if $U - \lambda V = 0$, λ, is one of the roots of the cubic obtained by taking the discriminant of $U - \lambda V$, namely,

$$\Delta - \Theta\lambda + \Theta'\lambda^2 - \Delta'\lambda^3 = 0. \tag{11}$$

We thus obtain

$$\tan\theta = \frac{\dfrac{d\phi}{dx}\dfrac{d\psi}{dy} - \dfrac{d\phi}{dy}\dfrac{d\psi}{dx}}{\dfrac{d\phi}{dx}\dfrac{d\psi}{dx} + \dfrac{d\phi}{dy}\dfrac{d\psi}{dy}} = \frac{(\lambda a' - a)(\lambda b' - b) - (\lambda h' - h)^2}{(\lambda h' - h)\{(a' + b')\lambda - (a + b)\}}. \tag{12}$$

Suppose $h = h' = 0$, then the two cubics corresponding to the two rectangular directions parallel to the axes of coordinates cut at right angles at the vertices of the common self-conjugate triangle of the conics. In this case one of the conics, V, say, is a circle, and the axes of all the conics of the system are parallel to the axes of coordinates. Thus the two cubics in this case are the loci of the vertices of a system of conics passing through four points on a circle. I $V = x^2 + y^2 - k^2$, these loci are

$$(a - b)xy^2 + g(y^2 - x^2) - 2fxy - k^2(ax + g) - cx = 0, \tag{13}$$

$$(a - b)x^2y + f(y^2 - x^2) + 2gxy + k^2(by + f) + cy = 0, \tag{14}$$

so that they touch infinity and meet it again in a rectangular direction. We have thus proved that the cubics (13) and (14) cut at right angles at their seven finite points of intersection.

Now, suppose $a+b=a'+b'=0$ in (12), then, as in the preceding case the cubics ϕ, ψ cut at right angles at the three points. All the conics passing through the four points are then equilateral hyperbolæ, so that one of the points is the intersection of the perpendiculars of the triangle formed by the other three points. Considering then a triangle formed by three points of intersection of U and V, we see that the cubics intersect at the intersection of the perpendiculars and the feet of the perpendiculars, besides the three points themselves. Again, for the two points at infinity common to the cubics, we have, from (10),

$$(ax+hy)(h'x-a'y)-(hx-ay)(a'x+h'y)=0$$

or
$$(ah'-a'h)(x^2+y^2)=0\,; \tag{15}$$

that is, the two cubics are circular. But two circular cubics cutting each other at right angles at their seven finite points of intersection are confocal. Hence, we see that the two curves ϕ, ψ are in this case confocal circular cubics. Suppose we consider now, in the same case, two cubics corresponding to directions which are not at right angles to each other. For the angles of intersection of $l\psi-m\phi$, $l'\psi-m'\phi$ at the feet of the perpendiculars of the triangle formed by three points of intersection of the equilateral hyperbolæ, U, V, we have, instead of (12),

$$\tan\theta=\frac{(lm'-l'm)\left(\frac{d\phi}{dx}\frac{d\psi}{dy}-\frac{d\phi}{dy}\frac{d\psi}{dx}\right)}{ll'\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2\right\}+mm'\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2\right\}-(lm'+l'm)\left\{\frac{d\phi}{dx}\,d\psi+\frac{d\phi}{dy}\frac{d\psi}{dy}\right\}}.$$

But from (7) and (8) we have

$$\frac{d\phi}{dx}+\frac{d\psi}{dy}=0,\quad \frac{d\phi}{dy}-\frac{d\psi}{dx}=0\,,$$

which gives
$$\tan\theta=\frac{lm'-l'm}{ll'+mm'}\,; \tag{16}$$

that is, the angle θ is equal to the angle between the real asymptotes of the circular cubics. The two circular cubics thus cut at the same angle at all their seven finite points of intersection. It may be observed that if S_1, S_2, S_3 are the circles

described on the sides of the triangle formed by three points of intersection of the equilateral hyperbolæ as diameters, one of the cubics can be written

$$lS_2S_3 + mS_3S_1 + nS_1S_2 = 0, \tag{17}$$

where $l + m + n = 0$, for this represents a circular cubic passing through the vertices, the intersection of the perpendiculars and the feet of the perpendiculars.

Let us now consider the locus of the points of contact of tangents drawn from a fixed point to the system of conics passing through four fixed points. If Q, P are the polars of the fixed point with regard to the conics U, V respectively, the locus is found to be the cubic

$$PU - QV = 0. \tag{18}$$

This cubic, like the preceding ones, passes through the four points $U = 0$, $V = 0$, and the intersections of diagonals and opposite sides of the quadrangle formed by the four points, namely—the three latter points—the vertices of the common self-conjugate triangle of the conics U, V. Let us now consider the angles of intersection of cubics corresponding to two different points at the seven points of intersection that we have just mentioned. Let the axes be taken so that the two points are $\pm k$, 0, then $P = N' + kL'$, $P' = N' - kL'$, $Q = N + kL$, $Q' = N - kL$.

For one of the four points given by $U = V = 0$, we get

$$\cot\theta = \frac{(N'^2 - k^2L'^2)(L^2 + M^2) + (N^2 - k^2L^2)(L'^2 + M'^2) - 2(NN' - k^2LL')(LL' + MM')}{2k(LN' - L'N)(LM' - L'M)}$$

$$= \frac{(LN' - NL')^2 + (MN' - M'N)^2 - k^2(LM' - L'M)^2}{2k(LN' - L'N)(LM' - L'M)};$$

but we have $\quad Lx + My + N = 0, \quad L'x + M'y + N' = 0,$

from $U = 0$, $V = 0$, and from these we obtain

$$\frac{x}{MN' - MN'} = \frac{y}{NL' - N'L} = \frac{1}{LM' - L'M},$$

so that we get

$$\cot\theta = \frac{x^2 + y^2 - k^2}{2ky}; \tag{19}$$

that is, the angle of intersection at 0, one of the four points $U = V = 0$, is equal to the angle subtended at 0 by the points $\pm k$, 0, A, B, say. This is also apparent from the fact that the tangents to the cubic (18) at the four points all pass through the point A, which is on the curve, a result that can be proved by projecting what we have already arrived at with regard to the locus of the points of contact of parallel tangents. Hence two cubics such as (18) corresponding to two points A, B will cut orthogonally at such of the four points as are made to lie on the circle described on AB as diameter; for instance, if we have a system of conics passing through four points on a circle, two cubics corresponding to the points A, B at the extremities of a diameter of the circle will cut orthogonally at the four points. I now seek the angles of intersection of the cubics at the vertices of the common self-conjugate triangle of the conics U, V.

Putting $V = \lambda U$, where λ is determined by the cubic (11), we easily find

$$\cot\theta = \frac{(\lambda g - g')^2 + (\lambda f - f')^2 - k^2(\lambda a - a')^2 - k^2(\lambda h - h')^2}{2k\{(\lambda a - a')(\lambda f - f') - (\lambda g - g')(\lambda h - h')\}}. \tag{20}$$

This expression simplifies if the two points lie on the locus of the centres of the conics of the system, namely, the conic passing through the middle points of all the lines joining the four points. We may then take U so as to have its centre at k, 0 and V its centre at $-k$, 0. This gives $ka + g = 0$, $kh + f = 0$, $-ka' + g' = 0$, $-kh' + f' = 0$, whence

$$g^2 + f^2 = k^2(a^2 + h^2), \quad af - gh = 0, \tag{21}$$

$$g'^2 + f'^2 = k^2(a'^2 + h'^2), \quad a'f' - g'h' = 0, \tag{22}$$

so that (20) becomes

$$\cot\theta = \frac{aa' + hh'}{a'h - ha'}, \tag{23}$$

or

$$\frac{gg' + ff'}{f'g - fg'}, \tag{24}$$

that is, the angles of intersection at the three points are the same and equal to the angle between the polars of the origin with regard to the two particular conics U, V, of the system. In this case the circle circumscribing the common self-conjugate triangle passes through the tangentials A', B' of A, B with regard to the two cubics, respectively, for the tangents at the vertices to the cubics are concurrent at A', B', respectively. Hence, if A', B' are at the extremities of a diameter of the circle, the cubics cut orthogonally at the vertices of the triangle.

I have noticed that in the case of a system of conics passing through four concyclic points the locus of the vertices consists of two cubics. The locus of the foci in the same case, as has been shown by Sylvester, consists of the two circular cubics having the four points for foci. It is thus suggested that the locus of points on the major axis of the conics, at a distance x from the centre given by $x^2 = a^2 + kb^2$, where a, b are the semiaxes and k is a constant, is also a cubic, and this, in fact, is found to be the case. Let S, S' denote the two cubics (13) and (14), then the equations of two such loci corresponding to the axes of x and y, respectively, are found to be

$$S' + kY = 0, \; S + k'X = 0, \tag{25}$$

where Y and X are, respectively, the equations of the three parallels to the axes of x and y drawn through the vertices of the common self-conjugate triangle of the conics. The confocal circular cubics, of course, correspond to $k = k' = -1$, and by taking $k = k' = 1$, we get the locus of the points where the director circles are met by the axes of the conics.

I now consider the problem of describing a cubic through six points on a conic, so that the tangents at these points all pass through a point P, that is, so that the conic is the polar conic of the point P with regard to the cubic. If we consider two solutions of this problem and P, Q are the two corresponding points the angle of intersection at one of the six points O is evidently the angle subtended at O by PQ. Also if P, Q be taken on the line at infinity, the angles of intersection at all the six points are equal to the angle between the directions P, Q. Let x, y be the tangents from P to the conic, and z the chord of contact, then the conic may be taken as $z^2 - xy = 0$, and the cubic is easily seen to be of the form

$$U = z^3 - 3xyz + ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0. \tag{26}$$

For the six points of intersection of the cubic and the conic, putting $x = 1$, $y = \theta^2$, $z = \theta$, we get

$$d\theta^6 + 3c\theta^4 - 2\theta^3 + 3b\theta^2 + a = 0, \tag{27}$$

in which it is to be observed that the coefficients of θ^5 and θ are absent. Now ∞, 0 are the values of θ corresponding to the tangents x, y drawn from P to the conic. Thus we see that these values are such that the corresponding factors being taken as lines of reference the binary sextic giving the six points on the

conic wants its second and penultimate terms. Now if we have a binary sextic $U = (a_0, a_1, a_2, a_3, a_4, a_5, a_6)(x, y)^6$ and transform to a new set of variables and make the coefficients of the second and penultimate terms vanish we get

$$x' \frac{du}{dx} + y' \frac{du}{dy} = 0, \quad x \frac{du'}{dx'} + y \frac{du'}{dy'} = 0, \tag{28}$$

from which by elimination of x', y' we get an equation in x, y of the 26th degree; but this must be divisible by U as (28) are satisfied by $xy' - yx' = 0$, $u = u' = 0$. Thus the result of elimination is of the twentieth degree, that is, it consists of ten quadratic factors. We thus see that there are ten cubics and ten points such as P. Let us now consider two of the cubics, taking the lines joining P, Q and the pole of PQ with regard to the conic as triangle of reference. Let the conic be

$$U = ax^2 + by^2 + cz^2 + 2hxy, \tag{29}$$

then if U_1, U_2 are the two cubics, $\frac{dU_1}{dx}$ and $\frac{dU_2}{dy}$ must be proportional to U. Hence integrating we get

$$\begin{aligned} U_1 &= 3xU - (2ax^3 + 3hx^2y) + \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3, \\ U_2 &= 3yU - (2by^3 + 3hxy^2) + \alpha' x^3 + \beta' x^2 z + \gamma' x z^2 + \delta' z^3; \end{aligned} \tag{30}$$

but U_1 and U_2 intersect on U, so that we must have

$$\begin{aligned} l(\alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3) - l(2ax^3 + 3hx^2 y) \\ + m(\alpha' x^3 + \beta' x^2 z + \gamma' x z^2 + \delta' z^3) - m(2by^3 + 3hxy^2) \\ = (\lambda x + \mu y + \nu z)(cz^2 + ax^2 + by^2 + 2hxy) \end{aligned} \tag{31}$$

from which, by comparison of the coefficients of z, we obtain

$$l\beta y^2 + m\beta' x^2 = \nu(ax^2 + by^2 + 2hxy),$$

which gives $\beta = \beta' = \gamma = 0$. Hence as the cubics intersect again at three points on the line $\lambda x + \mu y + \nu z$, which takes the form $\lambda x + \mu y = 0$, we see that the cubics U_1, U_2 intersect again at three points lying on the line passing through the pole of PQ with regard to the conic. Let z be at infinity and x, y rectangular coordinates then U_1, U_2 cut orthogonally at six points on U.

In the case in which the tangents at four points on the cubic pass through a point not on the curve, then if $x = l\alpha$, $y = m\beta$, $z = n\gamma$, where α, β, γ are the

perpendiculars from a point on the sides of the triangle of reference, we may take $x^2 = y^2 = z^2$ as the four points, and then if a, b, c is the intersection of the tangents thereat, the cubic takes the form

$$a^2(by + cz)(y^2 - z^2) + b^2(cz + ax)(z^2 - x^2) + c^2(ax + by)(x^2 - y^2) + \theta\{ax(y^2 - z^2) + by(z^2 - x^2) + cz(x^2 - y^2)\} = 0, \quad (32)$$

where θ is arbitrary. The polar conic of a, b, c is, it may be observed,

$$(3a^2 + \theta)(b^2 - c^2)x^2 + (3b^2 + \theta)(c^2 - a^2)y^2 + (3c^2 + \theta)(a^2 - b^2)z^2 = 0, \quad (33)$$

which, of course passes through the four points and also meets the cubic again at two points lying on the line

$$3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)(ax + by + cz) + ax(b^2 - c^2)(\theta + 3a^2)(\theta - a^2) + by(c^2 - a^2)(\theta + 3b^2)(\theta - b^2) + cz(a^2 - b^2)(\theta + 3c^2)(\theta - c^2) = 0, \quad (34)$$

which, when θ varies, envelopes a conic.

Hence, if we take two cubics, such as (32), corresponding to two points, P, P', the angle of intersection at one of the four points, O say, is equal to the angle subtended at O by PP'.

I observe here that if a cubic be referred to an inscribed triangle, the tangents at whose vertices pass through a point a, b, c, the equation of the curve can be written

$$bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) + 2mxyz = 0, \quad (35)$$

in which case we see that the lines joining the vertices to the points in which the opposite sides respectively meet the curve again, pass through a point, namely, bc, ca, ab. If $m = 0$ in (35), we have a cubic already considered, namely, that on which the four points $x^2 = y^2 = z^2$ are such that the tangents thereat intersect in the point bc, ca, ab on the curve, and the tangents at the vertices of the triangle of reference, namely, the intersections of diagonals and opposite sides of the quadrangle formed by the four points, intersect at the point a, b, c, the tangential of bc, ca, ab. If these cubics are both circular, it may be observed that the *satellites* ε, ζ are either parallel or at right angles to each other.

Again, let α, β, γ, δ denote the lengths of the perpendiculars from a point on four lines, then it is readily seen that two cubics cutting each other at right angles at the three collinear points $\alpha\delta$, $\beta\delta$, $\gamma\delta$ can be written

$$\left.\begin{aligned}(\alpha + \delta)(\beta + \delta)(\gamma + \delta) - k\delta^2\varepsilon = 0,\\ (\alpha - \delta)(\beta - \delta)(\gamma - \delta) - k'\delta^2\zeta = 0.\end{aligned}\right\} \quad (36)$$

We can hence show that the cubics

$$(a+\lambda a')\,x\,(my^2-nz^2)+(b+\lambda b')\,y\,(nz^2-lx^2)$$
$$+(c+\lambda c')\,z\,(lx^2-my^2)+2pxyz=0, \quad (37)$$

where l, m, n, p are variable, are such that the tangents at the vertices of the triangle pass through a point. This point is given by $(a+\lambda a')\,x=(b+\lambda b')\,y=(c+\lambda c')\,z$, so that if λ is variable, it lies on the conic

$$(bc'-b'c)\,yz+(ca'-c'a)\,zx+(ab'-a'b)\,xy=0. \quad (38)$$

If this conic is made to coincide with the circumscribing circle, namely, if the points bc, ca, ab; $b'c'$, $c'a'$, $a'b'$ lie on the circumscribing circle, two cubics intersect at the vertices at an angle equal to that subtended by the two points at any point of the circle.

I now mention a case in which two cubics intersect orthogonally at six points of their intersection. Let u, v be circular coordinates; that is, let $x+iy=u$, $x-iy=v$, where x, y are rectangular coordinates, then let us consider the quartic curve

$$\phi=a\,(u^4+v^4)+b\,(u^3+v^3)+c\,(u^2+v^2)+lu^2v^2+muv$$
$$+\alpha uv\,(u^2+v^2)+\beta uv\,(u+v). \quad (39)$$

Now, let us take ϕ, so that $\frac{d\phi}{du}-\frac{d\phi}{dv}$ is proportional to $(u-v)\frac{d^2\phi}{du\,dv}$. We thus get

$$4a\,(u^2+v^2+uv)+3b\,(u+v)+2c=k\{4luv+m+3\alpha\,(u^2+v^2)+2\beta\,(u+v)\}, \quad (40)$$

whence $\alpha=\frac{4a}{3k+1}$, $\beta=\frac{3b}{2k+1}$,

$$l=\frac{6a\,(k+1)}{(2k+1)(3k+1)},\ m=\frac{2c}{k+1}. \quad (41)$$

It then readily follows that the cubics $\frac{ld\phi}{dx}+\frac{md\phi}{dy}$, $\frac{l'd\phi}{dx}+\frac{m'd\phi}{dy}$, where $ll'+mm'=0$ or, which is the same thing, the cubics $\frac{d\phi}{du}\pm\frac{\lambda d\phi}{dv}$, cut orthogonally at six points on the conic $\frac{d^2\phi}{dudv}$ or $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}$, namely,

$$4a\,(u^2+v^2+uv)+3b\,(u+v)+2c=0,$$

or
$$4a(3x^2 - y^2) + 3bx + c = 0. \tag{42}$$

Let us now consider a cubic and a conic cutting each other orthogonally at their six points of intersection. Since a cubic and a conic together involve 14 constants and 6 conditions are involved in the orthogonality, there only remain 8 constants, which thus are one too few in number to give a general cubic involving nine constants. It hence appears that a cubic must satisfy one relation with the circular points at infinity if it is capable of being cut orthogonally by a conic at six points. I now propose to determine such a cubic when the conic is given.

Let the conic referred to its principal axes be $\alpha x^2 + \beta y^2 + 1$, then if the cubic is ϕ, we must evidently have

$$\alpha x \frac{d\phi}{dx} + \beta y \frac{d\phi}{dy} + k\phi = (lx + my + n)(\alpha x^2 + \beta y^2 + 1). \tag{43}$$

Now let $\phi = \phi_3 + \phi_2 + \phi_1 + \phi_0$, where ϕ_n is a rational integral homogeneous expression in x, y of the n^{th} degree, then we must have

$$\left.\begin{aligned} \alpha x \frac{d\phi_3}{dx} + \beta y \frac{d\phi_3}{dy} + k\phi_3 &= (lx + my)(\alpha x^2 + \beta y^2), \\ \alpha x \frac{d\phi_2}{dx} + \beta y \frac{d\phi_2}{dy} + k\phi_2 &= n(\alpha x^2 + \beta y^2), \\ \alpha x \frac{d\phi_1}{dx} + \beta y \frac{d\phi_1}{dy} + k\phi_1 &= lx + my, \\ k\phi_0 &= n. \end{aligned}\right\} \tag{44}$$

In the case of ϕ_2 there is an ambiguity, for if $\phi_2 = ax^2 + by^2 + 2hxy$, we have from the coefficient of xy, $h(\alpha + \beta + k) = 0$, which gives either $h = 0$ or $\alpha + \beta + k = 0$. In the first case the cubic is

$$\frac{l\alpha x^3}{3\alpha + k} + \frac{m\alpha x^2 y}{2\alpha + \beta + k} + \frac{l\beta xy^2}{2\beta + \alpha + k} + \frac{m\beta y^3}{3\beta + k} + \frac{n\alpha x^2}{2\alpha + k} + \frac{n\beta y^2}{2\beta + k} + \frac{lx}{\alpha + k} + \frac{my}{\beta + k} + \frac{n}{k} = 0, \tag{45}$$

and in the second case

$$\frac{l\alpha x^3}{2\alpha - \beta} + mx^2 y + lxy^2 + \frac{m\beta y^3}{2\beta - \alpha} + n\frac{(\alpha x^2 - \beta yz)}{\alpha - \beta} + 2hxy - \frac{lx}{\beta} - \frac{my}{\alpha} - \frac{n}{\alpha + \beta} = 0. \tag{46}$$

The latter cubic, it may be observed, can be written

$$lx\left\{\frac{ax^2}{2a-\beta}+y^2-\frac{1}{\beta}\right\}+my\left\{x^2+\frac{\beta y^2}{2\beta-a}-\frac{1}{a}\right\}$$
$$+n\left\{\frac{ax^2-by^2}{a-\beta}-\frac{1}{a+\beta}\right\}+2hxy=0, \qquad (47)$$

in which the three conics multiplied by lx, my and n, respectively, are all confocal with the given conic $ax^2+\beta y^2+1=0$. Again, if the cubic (45) be written

$$lxU_1+myU_2+nU_3=0, \qquad (48)$$

the three conics U_1, U_2, U_3 are confocal with $ax^2+\beta y^2+1$, corresponding to three values, $\frac{1}{2}(k+a)$, $\frac{1}{2}(k+\beta)$, $\frac{1}{2}k$, respectively, of λ in the equation

$$\frac{ax^2}{\lambda+a}+\frac{\beta y^2}{\lambda+\beta}+\frac{1}{\lambda}=0. \qquad (49)$$

Similarly in the case of the sphere we can show that the most general equation of a cubic cone cutting orthogonally the cone $ax^2+by^2+cz^2=0$ is

$$lxU_1+myU_2+nzU_3=0, \qquad (50)$$

where U_1, U_2, U_3 are the three cones confocal with the given one, which are obtained by taking λ equal to $\frac{1}{2}(3k-a)$, $\frac{1}{2}(3k-b)$, $\frac{1}{2}(3k-c)$ respectively, in

$$\frac{ax^2}{a-\lambda}+\frac{by^2}{b-\lambda}+\frac{cz^2}{c-\lambda}=0. \qquad (51)$$

Again, if $3k=a+b+c$, the most general equation of the orthogonal cubic cone is

$$lxU_1+myU_2+nzU_3+pxyz=0. \qquad (52)$$

As an extension of the preceding results I may notice that we can obtain in the same way the equation of a cubic surface cutting a given quadric orthogonally along the entire curve of intersection. If the cubic be written $\phi_3+\phi_2+\phi_1+\phi_0$, where ϕ_n is a homogeneous expression in x, y, z of the n^{th} degree, we have if the quadric is $ax^2+by^2+cz^2+d$,

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}\right)(\phi_3+\phi_2+\phi_1+\phi_0)+k(\phi_3+\phi_2+\phi_1+\phi_0)$$
$$=(lx+my+nz+p)(ax^2+by^2+cz^2+d). \qquad (53)$$

We thus obtain the equation of the cubic in the form

$$lxU_1 + myU_2 + nzU_3 + pU_4 = 0, \tag{54}$$

where U_1, U_2, U_3, U_4 are four confocal quadrics involving the parameter k. If k have a certain determinate value we may add to (54) the term $qxyz$, and further, if there is a relation connecting a, b, c, a term such as rxy or ryz or rzx may be added.

I now proceed to consider some properties of confocal nodal circular cubics. Such a curve involves six constants, so that when the two foci are given, there are still two indeterminate parameters. By inverting the equation of a conic from a point on itself, we get the equation of such a cubic in the form

$$PB\rho_1 \pm PA\rho_2 + (PA \pm PB)\rho = 0, \tag{55}$$

where A, B are the foci, P is the node, and ρ_1, ρ_2, ρ are the distances of a point from A, B, P, respectively.

It may be observed that A, B, may be imaginary, that is, A, B may be the antipoints of two real points, A', B', in which case, the points A', B' are foci, and the equation (55) gives another solution of the problem to describe a nodal circular cubic with given foci and node. Thus there are four solutions of the problem altogether. Now let us make use of elliptic coordinates, the points A, B being taken as the foci of the system of confocal conics, that is, let the conics be

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - c_2} = 1, \frac{x^2}{\nu^2} - \frac{y^2}{c^2 - \nu^2} = 1,$$

whence
$$cx = \mu\nu, \; cy = \sqrt{\{(\mu^2 - c^2)(c^2 - \nu^2)\}}. \tag{56}$$

Now let the node of (55) lie on the axis of y, then from (55) we have

$$\rho_1 + \rho_2 = 2\rho, \text{ or } \mu = \rho, \tag{57}$$

whence $\mu^2 = x^2 + (y - \beta)^2$, where 0, β are the coordinates of the node. From this we obtain

$$\sqrt{(c^2 - \nu^2)}\{\mu \pm \sqrt{(\mu^2 - c^2)}\} = c\beta, \tag{58}$$

whence
$$\frac{d\mu}{\sqrt{(\mu^2 - c^2)}} \mp \frac{\nu d\nu}{c^2 - \nu^2} = 0. \tag{59}$$

Again consider the cubic with a node on the axis of x, the points A, B in this case in (55) being the imaginary foci of the system of conics. We have then

$$\mu^2 - c^2 = \rho'^2 = (x-\alpha)^2 + y^2, \tag{60}$$

whence

$$\nu\{\mu \pm \sqrt{(\mu^2 - c^2)}\} = c\alpha, \tag{61}$$

and

$$\frac{d\mu}{\sqrt{(\mu^2 - c^2)}} \pm \frac{d\nu}{\nu} = 0. \tag{62}$$

Now if two curves are represented by the differential equations

$$Pd\mu + Qd\nu = 0,\ P'd\mu + Q'd\nu = 0,$$

and there is

$$PP'(\mu^2 - c^2) + QQ'(c^2 - \nu^2) = 0, \tag{63}$$

then these two systems of curves cut orthogonally. But this condition is satisfied for (59) and (62), so that the systems of cubics we have been considering cut orthogonally. This orthogonality occurs at only one point of intersection of the cubics, which, we see from (57) and (60), intersect on the line $\rho^2 - \rho'^2 = c^2$. It may be observed that the remaining four points of intersection of the two cubics lie on a point circle, for from (57) we have $\mu = \rho$, and for (60), we may substitute $\nu^2 - c^2 = \rho'^2$, whence $\mu^2 + \nu^2 - c^2 = \rho^2 + \rho'^2$, that is, $(x-\alpha)^2 + (y-\beta)^2 = 0$.

In the general case when the node of the cubic is at an arbitrary point, its equation in elliptic coordinates can be written

$$\{r\cos\alpha + \sin\alpha\,\nu(c^2 - \nu^2)\}\{\mu \pm \nu(\mu^2 - c^2)\} = cp, \tag{64}$$

or, in x, y coordinates

$$x^2 + y^2 - 4p(x\cos\alpha + y\sin\alpha) + 2p^2 + c^2\cos 2\alpha = \rho\rho' = \pm\sqrt{\{(x^2+y^2)^2 - 2c^2(x^2 - y^2) + c^4\}}. \tag{65}$$

Now when two curves are represented by $Pd\mu + Qd\nu = 0$, $P'd\mu + Q'd\nu = 0$, the angle θ between them is given by

$$\tan\theta = \frac{(PQ' - P'Q)\,\nu\{(\mu^2 - c^2)(c^2 - \nu^2)\}}{PP'(\mu^2 - c^2) + QQ'(c^2 - \nu^2)}. \tag{66}$$

Hence for the cubic (64) and a similar cubic with β instead of α we find $\theta = \alpha - \beta$, that is, the angle at one point of intersection is equal to the angle between the asymptotes, for from (65) we see that $x\cos\alpha + y\sin\alpha = 0$ is parallel to the real asymptote of the curve. This point is one of three points of intersection lying on

a line, as we see from (65), from which also we see that the other four points of intersection lie on a circle.

With respect to the system of circular cubics

$$lx(x^2+y^2+c^2)+my(x^2+y^2-c^2)+n(x^2+y^2)=0, \tag{67}$$

where l, m, n are variable, I observe that they all have the origin, which they pass through, as a double focus. Hence two of the system intersect elsewhere in four points, two points being real and two imaginary; the two latter being the antipoints of the former two.

Since (67) can be written in the form

$$\lambda\left(u+\frac{c^2}{u}\right)+\mu\left(v+\frac{c^2}{v}\right)+\nu=0,$$

where $x+iy=u$, $x-iy=v$, we can easily deduce that two cubics of the system (67) cut each other at all their five finite points of intersection at an angle equal to that between their asymptotes

$$lx+my+n=0, \quad l'x+m'y+n'=0.$$

With regard to the system of equilateral hyperbolæ

$$xy-ab-m(bx-ay)=0, \tag{68}$$

passing through the fixed points $\pm a$, $\pm b$, I observe that they are cut at right angles, at all the six points of intersection, by the system of cubics

$$\frac{x^3}{a}+\frac{y^3}{b}-3(ax+by)=c'. \tag{69}$$

I now proceed to consider some cases of conics cutting each other orthogonally at certain points of their intersection. First of all, I shall obtain a locus passing through all the points of intersection of two conics at which they cut orthogonally. Let $(a, b, c, f, g, h)(x, y, 1)^2=0$ be the equation of a conic U in rectangular Cartesian coordinates, and let $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2=0$ be the corresponding tangential equation, then it is easy to see that

$$S=C(x^2+y^2)-2Gx-2Fy+A+B=0 \tag{70}$$

gives the director circle, viz., the locus of intersection of rectangular tangents of U. Hence, the director circle of $U_1 + \lambda U_2$ is

$$S_{11} + \lambda S_{12} + S_{22} = 0, \tag{71}$$

where

$$\begin{aligned} S_{12} = (a_1b_2 + b_1a_2 - 2h_1h_2)(x^2 + y^2) - 2x(f_1h_2 + f_2h_1 - b_1g_2 - b_2g_1) \\ - 2y(g_1h_2 + g_2h_1 - a_1f_2 - a_2f_1) + c_1(a_2 + b_2) \\ + c_2(a_1 + b_1) - 2g_1g_2 - 2f_1f_2. \end{aligned} \tag{72}$$

The circle S_{12} is in fact the director circle of the covariant conic Φ, namely, the envelope of lines divided harmonically by U_1, U_2. Now, suppose the conics U_1, U_2 to cut orthogonally at a point P, then taking the origin at this point, we have $c_1 = c_2 = 0$, $g_1g_2 + f_1f_2 = 0$, and when these relations are satisfied, we see from (72) that S_{12} also passes through P. It may be observed that the equation of the director circle in circular coordinates is

$$S = Cxy - Gy - Fx + H = 0, \tag{73}$$

and that of S_{12} is

$$C_{12}xy - G_{12}y - F_{12}x + H_{12} = 0. \tag{74}$$

Hence, if U_1, U_2 cut orthogonally at all their four points of intersection, we must have $lU_1 + mU_2 = S_{12}$, or the circle S_{12} must vanish identically. Taking U_2 referred to its principal axes, we have

$$\begin{aligned} (a_1b_2 + b_1a_2)(x^2 + y^2) + 2b_2g_1x + 2a_2f_1y + c_1(a_2 + b_2) + c_2(a_1 + b_1) \\ = l(a_1x^2 + b_1y^2 + 2h_1xy + 2g_1x + 2f_1y + c_1) + m(a_2x^2 + b_2y^2 + c_2), \end{aligned}$$

whence, by comparison of the coefficients of xy, x and y, we obtain

$$lh_1 = 0, \quad lg_1 = b_2g_1, \quad lf_1 = a_2f_1. \tag{75}$$

Taking $f_1 = g_1 = h_1 = 0$, we come upon the case of confocal conics. Again, let $l = 0$, in which case S_{12} vanishes identically; that is, $f_1 = g_1 = 0$, and h_1 remains indeterminate. In this case, if U_2 is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, U_1 is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{(a^2 - b^2)}{a^2 + b^2} + 2hxy = 0, \tag{76}$$

in which it is to be observed that the conic corresponding to $h=0$ is a confocal given by $a'^2b^2+b'^2a^2=0$. The equation (76) evidently denotes a system of conics passing through the vertices of a rhombus, which is real if the conic U_2 is a hyperbola.

Again from (75), we can take $h_1=0$, $l=a_2$ and $g_1=0$. This gives the system of conics

$$\frac{x^2}{a^2-2b^2}-\frac{y^2}{b^2}-1+2fy=0 \tag{77}$$

orthogonal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, the conic corresponding to $f=0$ being a confocal given by $b'^2+b^2=0$. In this case, it may be observed, the four points of intersection lie on a circle passing through the four foci. In the same way we get the orthogonal system

$$\frac{x^2}{a^2}+\frac{y^2}{2a^2-b^2}+1+2gx=0. \tag{78}$$

If we want to find the condition that two conics should cut orthogonally at one point of intersection, we have to express that U_1, U_2 and S_{12} have a point in common. This gives (see Salmon's Conics) a relation of the form $T^2=64M$. This will be found to involve the coefficients of each of the conics U_1, U_2 in the eighth degree. It may be observed that the Jacobian of U_1, U_2, S_{12} gives a locus which passes through all the points at which U_1, U_2 cut orthogonally. Referring U_1, U_2 to their common self-conjugate triangle in trilinear coordinates, we have

$$U_1=l_1\alpha^2+m_1\beta^2+n_1\gamma^2,$$
$$U_2=l_2\alpha^2+m_2\beta^2+n_2\gamma^2,$$

and if $A=m_1n_2-m_2n_1$, $B=n_1l_2-n_2l_1$, $C=l_1m_2-l_2m_1$,

$$A'=m_1n_2+m_2n_1,\ B'=n_1l_2+n_2l_1,\ C'=l_1m_2+l_2m_1,$$

we obtain

$$J=A'\alpha\cos A(B\gamma^2+C\beta^2)+B'\beta\cos B(C\gamma^2+A\alpha^2)+C'\gamma\cos C(A\beta^2+B\alpha^2)$$
$$+\{A'(B+C)+B'(C+A)+C'(A+B)\}\alpha\beta\gamma=0, \tag{79}$$

this is the Hessian of another cubic, viz.

$$V=R(BCB'C'\alpha^3\cos B\cos C+CAC'A'\beta^3\cos C\cos A+ABA'B'\gamma^3\cos A\cos B)$$
$$-ABC(\alpha B'C'\cos B\cos C+\beta C'A'\cos C\cos A+\gamma A'B'\cos A\cos B)^3=0, \tag{80}$$

where

$$R = AB'^2C'^2\cos^2 B\cos^2 C + BC'^2A'^2\cos^2 C\cos^2 A + CA'^2B'^2\cos^2 A\cos^2 B$$
$$- A'B'C'\cos A\cos B\cos C\{A'(B+C)+B'(C+A)+C'(A+B)\}, \quad (81)$$

which represents another locus passing through all the points at which U_1, U_2 cut orthogonally. The conics U_1, U_2, S_{12} are polar conics of this cubic V.

It is to be observed that if U_1, U_2 cut orthogonally at two points, then U_1, U_2, S_{12} have two points in common and their Jacobian consists of a conic and a line intersecting in the two points, as we see by considering the Jacobian of three circles. Hence, in this case, we have

$$\frac{A'^2}{A}\cos^2 A = \frac{B'^2}{B}\cos^2 B,$$
$$\{A'(B+C)+B'(C+A)+C'(A+B)\}^2 = 4ABC'^2\cos^2 C'\}, \quad (82)$$

in which case J is divisible by

$$\frac{A'\alpha\cos A}{A} + \frac{B'\beta\cos B}{B} = 0, \quad (83)$$

the line on which U_1, U_2 cut orthogonally. Of course there are two other similar pairs of relations obtained by interchanging α, β, γ, etc. If U_1, U_2 cut orthogonally at three points, we have, in addition to (82),

$$\frac{B'^2}{B}\cos^2 B = \frac{C'^2}{C}\cos^2 C,$$

in which case

$$J = (\lambda\alpha+\mu\beta)(\mu\beta+\nu\gamma)(\nu\gamma+\lambda\alpha), \quad (84)$$

where $\lambda = \dfrac{A'\cos A}{A}$, $\mu = \dfrac{B'\cos B}{B}$, $\nu = \dfrac{C'\cos C}{C}$.

In this case it can readily be shown that the three points of intersection of U_1, U_2 lie on the circular cubics

$$\frac{\alpha}{\cos A}(\beta^2+\gamma^2+2\beta\gamma\cos A) \pm \frac{\beta}{\cos B}(\gamma^2+\alpha^2+2\gamma\alpha\cos B)$$
$$\pm \frac{\gamma}{\cos C}(\alpha^2+\beta^2+2\alpha\beta\cos C) = 0, \quad (85)$$

which curves, it is easy to see, are the loci of the foci of conics inscribed in the triangle and touching the radical axis of the circumscribed and nine-points circle

and the lines joining the feet of the perpendiculars respectively. Each of the three points lies on one of three properly selected cubics of the form comprised in (85). For a locus passing through all three points, we have, from (84), putting $A = A'^2 \cos^2 A$, $B = B'^2 \cos^2 B$, $C = C'^2 \cos^2 C$,

$$J = \left(\frac{\alpha}{A' \cos A} + \frac{\beta}{B' \cos B} + \frac{\gamma}{C' \cos C}\right)(A' \cos A\beta\gamma + B' \cos B\gamma\alpha + C' \cos C\alpha\beta) - \alpha\beta\gamma. \quad (86)$$

Now, let L be the polar of the intersection of the perpendiculars of the self-conjugate triangle with regard to the covariant conic Φ, namely,

$$\Phi = B'C'\alpha^2 + C'A'\beta^2 + A'B'\gamma^2,$$

then

$$L = \frac{\alpha}{A' \cos A} + \frac{\beta}{B' \cos B} + \frac{\gamma}{C' \cos C} \quad (87)$$

and

$$J(L, U_1, U_2) = A' \cos A\beta\gamma + B' \cos B\gamma\alpha + C' \cos C\alpha\beta, \quad (88)$$

and we get $J(U_1, U_2, S_{12})$ in the form

$$LJ(L, U_1, U_2) - kJ(U_1, U_2, \Phi), \quad (89)$$

where k can be determined in terms of invariants of U_1, U_2 and the circular points. This cubic, of course, is another locus in addition to the circle

$$S_{12} = A'(\beta^2 + \gamma^2 + 2\beta\gamma \cos A) + B'(\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B) + C'(\alpha^2 + \beta^2 + 2\alpha\beta \cos C). \quad (90)$$

Again, from geometrical considerations, we can obtain a circular cubic passing through all the points at which U_1, U_2 cut orthogonally. If, from any point P, tangents are drawn to the system of conics inscribed in a quadrilateral, they belong to a system in involution, of which the tangents to the two conics of the system which pass through P are the double lines. Hence, if the latter pair of lines are at right angles to each other, the tangents from P to one conic of the system must pass through the circular points at infinity, or, in other words, P must be a focus of that conic. Thus we see that the circular cubic, which is the locus of the foci of conics inscribed in the quadrilateral formed by the common tangents of U_1, U_2 passes through all the points at which U_1, U_2 cut orthogonally. Taking circular coordinates, the tangents drawn from the circular points

to the system of conics $\Sigma_1 + \lambda\Sigma_2$ touching the common tangents of ε_1, ε_2 are, respectively,

$$\left.\begin{aligned} C_1x^2 - 2G_1x + A_1 + \lambda(C_2x^2 - 2G_2x + A_2) = 0, \\ C_1y^2 - 2F_1y + B_1 + \lambda(C_2y^2 - 2F_2y + B_2) = 0, \end{aligned}\right\} \quad (91)$$

and the locus of the foci is

$$\begin{aligned}(C_1x^2 - 2Gx_1 + A_1)(C_2y^2 - 2F_2y + B_2) \\ - (C_2x^2 - 2G_2x + A_2)(C_1y^2 - 2F_1y + B_1) = 0, \quad (92)\end{aligned}$$

which represents a circular cubic, as the coefficient of x^2y^2 vanishes. If we take the origin at the focus of the parabola touching the four common tangents, the latter curve may be written $F_2v(\lambda + \mu) + H_2\lambda\mu = 0$, and (92) then becomes

$$Cxy(x - y) - 2(G - F)xy + Ay - Bx = 0. \quad (93)$$

Combining this with S_{12} (74), we get the conic

$$\{(F_{12}y + G_{12}x - H_{12})\}C_1(x - y) + 2(F_1 - G_1)\} + C_{12}(A_1y - B_1x) = 0. \quad (94)$$

If we now express that this conic and U_1, U_2, and S_{12} are connected by a linear relation we evidently obtain the systems of conditions which must be satisfied if U_1, U_2 cut orthogonally at three points of their intersection. When U_1, U_2 are referred to their common self-conjugate triangle, the circular cubic (92) is

$$\begin{aligned}\left(\frac{Al_1l_2\alpha}{\sin A} + \frac{Bm_1m_2\beta}{\sin B} + \frac{Cn_1n_2\gamma}{\sin C}\right)(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) \\ = (\alpha \sin A + \beta \sin B + \gamma \sin C)(Al_1l_2\alpha^2 \cot A \\ + Bm_1m_2\beta^2 \cot B + Cn_1n_2\gamma^2 \cot C). \quad (95)\end{aligned}$$

Eliminating $\alpha\beta\gamma$ between this and J (79) and introducing the relations $\frac{\alpha^2}{A} = \frac{\beta^2}{B} = \frac{\gamma^2}{C}$ we get the line

$$\begin{aligned}2(Al_1l_2 + Bm_1m_2 + Cn_1n_2)(A'BC \cos A\alpha + B'CA \cos B\beta + C'AB \cos C\gamma) \\ + M\{\alpha \cos A(A^2l_1l_2 - B^2m_1m_2 - C^2n_1n_2) \\ + \beta \cos B(B^2m_1m_2 - C^2n_1n_2 - A^2l_1l_2) \\ + \gamma \cos C(C^2n_1n_2 - A^2l_1l_2 - B^2m_1m_2)\} = 0, \quad (96)\end{aligned}$$

where $$M = A'(B + C) + B'(C + A) + C'(A + B).$$

Hence, if U_1, U_2 cut orthogonally at two points, this line must be a chord of

intersection, and if they cut orthogonally at three points the three coefficients of the line must each vanish identically. Again, combining (95) with S_{12} (90) we obtain the conic, as at (94),

$$\left\{\frac{Al_1l_2\alpha}{\sin A}+\frac{Bm_1m_2\beta}{\sin B}+\frac{Cn_1n_2\gamma}{\sin C}\right\}\{(B'+C')\,\alpha\sin B\sin C + (C'+A')\,\beta\sin C\sin A+(A'+B')\,\gamma\sin A\sin B\} = (A'\sin^2A+B'\sin^2B+C'\sin^2C)\{Al_1l_2\alpha^2\cot A + Bm_1m_2\beta^2\cot B+Cn_1n_2\gamma^2\cot C\}. \quad (97)$$

If both the conics U_1, U_2 are parabolæ, the cubic (92) reduces to a circle, namely, as is well known, the circle circumscribing the triangle formed by their three finite common tangents, or, in other words, the nine-points circle of their common self-conjugate triangle. Hence, if the parabolæ cut orthogonally at two points, these points must be the intersections of the circle just mentioned and S_{12}. Again, if the parabolæ cut orthogonally at three points, these two circles must coincide. Now S_{12} (90) being of the form $A'\rho_1^2\sin^2A+B'\rho_2^2\sin^2B+C'\rho_3^2\sin^2C=0$, where ρ_1, ρ_2, ρ_3 are the distances of a point from the vertices of the common self conjugate triangle, respectively, is readily seen to cut orthogonally the circumscribing circle. Hence, the self-conjugate triangle must be such that the nine-points circle cuts orthogonally the circumscribing circle. This gives

$$r^2=2R^2, \quad (98)$$

where r is the radius of the polar circle and R that of the circumscribing circle, or

$$\sin^2A+\sin^2B\sin^2C=1. \quad (99)$$

The trilinear equations of the two parabolæ are then

$$\alpha^2\cot A\cot C+\beta^2\cot B\cot C+\gamma^2\pm\left(\frac{\alpha^2\sin B}{\sin A\sin C}-\frac{\beta^2\sin A}{\sin B\sin C}\right)=0, \quad (100)$$

where it is to be observed that the three points of intersection are the reciprocal points (i. e. such that $\alpha\alpha'=\beta\beta'=\gamma\gamma'$), of the intersections of parallels to the opposite sides drawn through the vertices.

We might also consider parabolæ cutting each other orthogonally at three points in the following manner. Let the curves be

$$\left.\begin{aligned}u^2+2lu+2mv+n&=0,\\ v^2+2l'u+2m'v+n'&=0,\end{aligned}\right\} \quad (101)$$

where $u = x\cos\alpha + y\sin\alpha$, $v = x\cos\alpha - y\sin\alpha$, then expressing that these cut orthogonally we obtain a hyperbola of the form

$$Auv + Bu + Cv + D = 0. \tag{102}$$

Now let the origin be taken at the centre of this hyperbola, then

$$B = C = 0, \text{ or } m + l\cos 2\alpha = 0,\ l' + m'\cos 2\alpha = 0. \tag{103}$$

We thus get $uv = k^2$, where $k^2 = lm'\cos 2\alpha \sin^2 2\alpha$.

Hence from (101) we must express that the equations

$$u^3 + 2lu^2 + nu + 2mk^2 = 0,$$
$$2l'u^3 + n'u^2 + 2m'k^2u + k^4 = 0,$$

coincide. We may thus take $l = b$, $m = -b\cos\theta$, $l' = -a\cos\theta$, $m' = a$, $n = n' = -4ab\cos\theta$, where $2\alpha = \theta$, and $\cos\theta = \sqrt{5} - 2$; that is, the parabolæ

$$\left.\begin{aligned} u^2 + 2b(u - v\cos\theta) - 4ab\cos\theta = 0, \\ v^2 + 2a(v - u\cos\theta) - 4ab\cos\theta = 0, \end{aligned}\right\} \tag{104}$$

where $u = x\cos\alpha + y\sin\alpha$, $v = x\cos\alpha - y\sin\alpha$, $\theta = 2\alpha$, and $1 + 2\sin\alpha = \sqrt{5}$, cut orthogonally at three of their points of intersection.

If a parabola be given and we wish to find another parabola cutting it orthogonally at three points, we may proceed as follows: taking circular coordinates we write the first curve $\sqrt{x} + \sqrt{y} = 1$, or

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0, \tag{105}$$

and the second

$$A\lambda^2 + B\mu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0, \tag{106}$$

in tangential coordinates. The circle passing through the foci is then

$$2(G - F)xy - Ay + Bx = 0. \tag{107}$$

Expressing that this must be identical with the circle S_{12}, namely,

$$C'_{12}xy - F_{12}x - G_{12}y + H_{12} = 0, \tag{108}$$

we obtain

$$F - G = 2\lambda,\ s + G + \lambda A = 0, \tag{109}$$

$$s + F^2 - \lambda B = 0,\ s = AB - H^2 = H(F + G) - AF - BG + FG. \tag{109}$$

Putting $F = \theta + \lambda$, $G = \theta - \lambda$, we get $A + B = 4\theta$, and hence we take $A = 2\theta + \phi$, $B = 2\theta - \phi$, so that we have

$$2\theta H = 2\theta^2 + \lambda\phi, \text{ and } 16\theta^4 + 4\theta^2(\lambda^2 - \phi^2) = \lambda^2\phi^2,$$

which gives $\lambda^2 + 4\theta^2 = 0$, the factor $4\theta^2 - \phi^2$ being neglected.

Thus the parabola (106) becomes

$$(2\theta+\phi)\lambda^2+(2\theta-\phi)\mu^2+2\lambda\mu(\theta+i\phi)+2\theta\mu\nu(1+2i)+2\theta\nu\lambda(1-2i)=0, \quad (110)$$

where $i^2=-1$. This represents a system of parabolæ having the axis in common with the parabola

$$3x^2+3y^2+10xy-2(x+y)+3+4i(y-x)(y+x-3)=0, \quad (111)$$

and touching this same parabola at the point of contact of a tangent inclined at an angle of 45° to the axis of the parabola (105). The curve (111), it may be observed, in Cartesian coordinates, is

$$(4x+2y)^2-4x-24y+3=0,$$

the parabola (105) at the same time being $4y^2=4x+1$. From the results just obtained, we see that two parabolæ cannot cut orthogonally at more than three points. If the parabola $y^2-px=0$ is cut orthogonally by a conic U at four points, U must be the ellipse $y^2+2x^2=a$, in which case two of the points are real and two imaginary. It may be observed that $y^2=x$ is cut orthogonally at three points by the conic

$$y^3+2x^2+2\lambda xy+\lambda^2(2x-1)+2\lambda(1+\lambda^2)y=0. \quad (112)$$

Again, I observe that if the angles of the triangle of reference in trilinear coordinates are such that $1+\cos A+\cos B+\cos C=0$, then the equation

$$(\beta^2-\gamma^2)^2\cos B\cos C+(\gamma^2-\alpha^2)^2\cos C\cos A+(\alpha^2-\beta^2)^2\cos A\cos B=0 \quad (113)$$

represents two equilateral hyperbolæ cutting each other orthogonally at the centres of the three circles escribed to the sides. In the case of conics cutting each other orthogonally at three points, we have, referring them to the triangle formed by the points,

$$\left.\begin{aligned} U_1&=l_1\beta\gamma+m_1\gamma\alpha+n_1\alpha\beta,\\ U_2&=l_2\beta\gamma+m_2\gamma\alpha+n_2\alpha\beta \end{aligned}\right\} \quad (114)$$

and

$$l_1l_2+m_1m_2=(l_1m_2+l_2m_1)\cos C,\quad m_1m_2+n_1n_2=(m_1n_2+m_2n_1)\cos A,$$
$$n_1n_2+l_1n_2=(l_1n_2+l_2n_1)\cos B, \quad (115)$$

Eliminating l_2, m_2, n_2, we get

$$(\cos B\cos C-\cos A)\,l(m^2+n^2)+(\cos C\cos A-\cos B)\,m(n^2+l^2)$$
$$+(\cos A\cos B-\cos C)\,n(l^2+m^2)+2(1-\cos A\cos B\cos C)\,lmn=0. \quad (116)$$

If l, m, n are tangential coordinates, this shows that the lines

$$\frac{\alpha}{l_1} + \frac{\beta}{m_1} + \frac{\gamma}{l_1} = 0, \quad \frac{\alpha}{l_2} + \frac{\beta}{m_2} + \frac{\gamma}{n_3} = 0 \tag{117}$$

passing through the points where the tangents of U_1, U_2 respectively, at the vertices, meet the opposite sides, are corresponding tangents of the curve of the third class (116), namely, tangents such that the line joining the points of contact touches the curve.

Again, if l, m, n are the trilinear coordinates of a point, U_1, U_2 are the polar conics of such points with regard to the triangle and (116) shows that the points are corresponding points of a cubic curve passing through the vertices, through the points where

$$\frac{\alpha}{\cos A - \cos B \cos C} + \frac{\beta}{\cos B - \cos A \cos C} + \frac{\gamma}{\cos C - \cos A \cos B} = 0 \tag{118}$$

meets the sides, and through the middle points of the perpendiculars. If one of the angles is right, (116) breaks up into factors. Taking $C = 90°$, we get the factors

$$n = l \sin A + m \sin B, \tag{119}$$

$$\frac{\sin A}{l} + \frac{\sin B}{m} - \frac{1}{n} = 0. \tag{120}$$

In the first case, the three points are the three vertices of a rectangle inscribed in the curve, and in the second case, the line $\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0$ is at right angles to the side γ opposite the right angle.

By combining the equation (116) with the condition $l_1 \cos A + m_1 \cos B + n_1 \cos C = 0$ that U_1 should be an equilateral hyperbola, we get the three equilateral hyperbolæ described through the vertices of the triangle so as to cut orthogonally a conic passing through these points. If the orthogonal conic is an equilateral hyperbola, that is if two equilateral hyperbolæ can cut each other orthogonally at the three points, the angles of the triangle must be such that

$$1 = \sin^2 A + \sin^2 B + \sin^2 C. \tag{121}$$

Again, combining the equation (116) with

$$\sqrt{(l \sin A)} + \sqrt{(m \sin B)} + \sqrt{(n \sin C)}, \tag{122}$$

the condition that U should be a parabola, we get the six parabolæ that can be

described through the three points to cut orthogonally other conics at three points. A few observations can be made with regard to conics having a focus in common. If the conics

$$x^2 + y^2 = (ax + by + k)^2, \quad x^2 + y^2 = (a'x + b'y + k)^2 \tag{123}$$

cut orthogonally at two points on the chord of intersection

$$(a + a')x + (b + b')y + 2k = 0, \tag{124}$$

the relation

$$aa' + bb' + 1 = 0 \tag{125}$$

is satisfied. In this case the latera recta of the conics are equal and the eccentricities are such that $ee' \cos\theta = 1$, where θ is the angle between the directrices. With regard to the angles of intersection of two such conics, I observe that from geometrical considerations we see that the angle of intersection at a point P is equal to half the angle subtended at P by the two other foci H, H'. Let the distances of a point from H, H' be ρ_1, ρ_2, and let α, β be the corresponding directrices, then the conics are $\rho_1^2 = e_1^2\alpha^2$, $\rho_2^2 = e_2^2\beta^2$. Suppose α, β at right angles to each other, then the four points of intersection lie on the circle

$$\frac{\rho_1^2}{e_1^2} + \frac{\rho_2^2}{e_2^2} = \rho^2, \tag{126}$$

where ρ is the distance of a point from the intersection O of α, β. Now, suppose that this circle passes through H, H', then $e_1 = \frac{c}{a}$, $e_2 = \frac{c}{b}$, where a, b, c are the sides of the triangle O, H, H'. Subject to these three conditions, the angle $HOH' = 30°$ and the conics intersect at all their points of intersection at an angle of 30°.

With regard to the concentric conics

$$ax^2 + by^2 + 2hxy + c = 0, \; a'x^2 + b'y^2 + 2h'xy + c' = 0, \tag{127}$$

I observe that if $a'b + b'a - 2hh' = 0$, or the asymptotes are harmonically connected, then

$$\cot\theta = \frac{c(a' + b') + c'(a + b)}{\sqrt{\{c'^2(ab - h^2) + c^2(a'b' - h'^2)\}}} \tag{128}$$

where θ is the angle of intersection at four points.

Again let as consider the angle of intersection of the parabolæ

$$\rho_1^2 = (x - \alpha)^2 + (y - \beta)^2 = x^2, \; \rho_2^2 = (x - \alpha')^2 + (y - \beta')^2 = y^2, \tag{129}$$

with rectangular directrices. These curves intersect on

$$\rho_1^2 + \rho_2^2 = R^2, \tag{130}$$

which is a circle passing through the foci, if the foci and the intersection of the directrices form an equilateral triangle. From geometrical considerations we have then $2\phi = \omega + \frac{\pi}{2}$, where ϕ is the angle of intersection at a point P, and ω is the angle subtended at P by the foci. But from (130) $\omega = 30°$, which gives $\phi = 60°$, that is, if two parabolæ are such that their directrices are rectangular and the triangle formed by the foci and the intersection of the directrices is equilateral, then they cut at an angle of 60° at their four points of intersection.

Estimate of Peirce's Linear Associative Algebra.

By H. E. Hawkes.

In the fourth volume of this journal appeared a remarkable memoir by Benjamin Peirce* in which the author made the first systematic attempt to classify and enumerate hyper-complex number systems. Though his work attracted wide and favorable comment in England† and America at the time, continental investigators on the subject during the last fifteen years have given him scarcely the credit which his results and his methods deserve. Adverse criticism has been due in part to a misunderstanding of Peirce's definitions, in part to the fact that certain of Peirce's principles of classification are entirely arbitrary and quite distinct in statement from those used by Study and Scheffers, in part to Peirce's vague and in some cases unsatisfactory proofs, and finally to the extreme generality of the point of view from which his memoir sprang, namely a "philosophic study of the laws of algebraic operation." If Peirce's work is to receive its due recognition, three questions must be discussed. *First.* What problem did Peirce attack, and to what extent did he solve it? *Second.* What relation does this problem bear to that treated by Study and Scheffers? *Third.* To what extent do Peirce's methods assist in the solution of that problem?

In this paper I propose to discuss the first and second questions. The third question the writer considers elsewhere.‡

Historical Review.

With the discovery and general use of the geometrical interpretation of the ordinary complex number, the question naturally arose whether a number system

* "Linear Associative Algebra," American Journal of Mathematics, vol. IV, p. 97, 1881. This memoir was read before the National Academy of Sciences in 1870 and a few lithographed copies were made in the same year.

† Spottiswoode, Proceedings of the London Mathematical Society, vol. IV, p. 152, 1872. Cayley, Collected Works, XII, p. 465; XI, p. 457.

‡ Transactions of the American Mathematical Society, vol. III.

could be found which was susceptible of an interpretation in space, somewhat analogous to that of the ordinary complex numbers in the plane. Gauss* felt the need of and attempted to form such a system. Whether he did or did not anticipate Hamilton in the discovery of quaternions,† his work in that direction has not until recently been published‡ and could have had no influence on Peirce's investigations. That Gauss was very familiar with certain properties of hypercomplex numbers is evident from his well-known query regarding the inadmissibility into arithmetic of number systems in more than two units, which stimulated Hankel,§ Weierstrass,‖ and many others to make investigations in this field.

In 1835 Hamilton¶ developed a complete theory of ordinary complex numbers in two units under the name of number couples. His definitions of complex numbers and the operations on them though ostensibly associated with "steps" of time have no immediate logical connection with the intuition** of time, and may be considered (essentially) as a discussion of pure complex numbers. De Morgan†† in his Double Algebra took the short step necessary to place complex numbers on a purely symbolic basis, in a form now very generally adopted‡‡. After defining equality and the operations of addition and subtraction of two complex numbers, Hamilton discusses§§ the operation of multiplication He defines the product

$$(0, b_1)(0, b_2) = (\gamma_{121} b_1 b_2 + \gamma_{122} b_1 b_2),$$

where γ_{121}, γ_{122} are arbitrary constants independent of b_1 and b_2. He proceeds to investigate the values that these "constants of multiplication" can have without introducing divisors of zero into the system, and determines on $\gamma_{121} = -1$, $\gamma_{122} = 0$ as the simplest possible values. This evidently leads him to the ordinary complex number system. This is, so far as I know, the first use of the

* Klein, Göttinger Nachrichten, 1898, p. 15; Tait, Proceedings of the Royal Society of Edinburgh, vol. XXIII.

† Life of Sir William Rowan Hamilton, by Graves, vol. III, p. 312.

‡ Werke, vol. VIII, p. 358.

§ "Theorie der Complexen Zahlensysteme," 1867, §29.

‖ Göttinger Nachrichten, 1884, p. 395.

¶ Transactions of the Royal Irish Academy, vol. XVII, p. 293.

** De Morgan pointed out this fact in the Transactions of the Cambridge Philosophical Society, vol. VII, p. 175, 1842.

†† "Trigonometry and Double Algebra," London, 1849.

‡‡ Burkhardt, "Analytische Functionen," pp. 1–9; Study, Encyklopädie der mathematischen Wissenschaften, vol. I, p. 149; Weber, "Lehrbuch der Algebra," Einleitung, p. 19.

§§ Hamilton, loc. cit., p. 400.

constants of multiplication of a number system. In 1843* Hamilton determines the values of the sixty-four constants of multiplication in quaternions. Hamilton does not at this time consider systems, in three units, but it appears† that as early as 1834 he was attempting to find a system in three units which would have a useful geometric interpretation, but was deterred by the presence of divisors of zero which he correctly believed existed in every such system. The fact that quaternions contained no divisors of zero Hamilton considered one of its chief merits.‡

In 1843§ Hamilton first discusses number system in n units. He defines operations on such numbers and gives the condition

$$\sum_{s=1}^{n} \gamma_{iks}\,\gamma_{sjt} = \sum_{s=1}^{n} \gamma_{kjs}\,\gamma_{ist}, \qquad (i, k, j, t = 1 \ldots . n) \quad (1)$$

that a system be associative.‖ In his treatment of division he does not arrive at divisors of zero explicitly, but contents himself with the remark that "the n sought coefficients of the symbolic quotient can *in general* be determined."¶ (The italics are mine.) Had he considered the case where the coefficients of the desired quotient could not be determined, he would have been lead to a discussion of divisors of zero. Hamilton attempted to develop the theory of, and perhaps to classify, systems in n units, but the existence of the n^4 equations (1) that the constants of multiplication of an associative system must satisfy seemed to offer insuperable difficulties.** Hamilton was considerably embarrassed in his early work†† by the lack of a useful definition of equality of two hypercomplex numbers. The assumption that

$$\sum_{1}^{n} a_i\, e_i = \sum_{1}^{n} b_i\, e_i$$

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* Transactions of the Royal Irish Academy, vol. XXI, p. 241.

† "Lectures on Quaternions," preface, p. 16, also p. 20.

‡ Life of Hamilton, vol. III, p. 254.

§ Transactions R. I. A., vol. XXI, p. 231.

‖ Ibid., p. 231; Schubert, (Encyc. math. Wiss., vol. I, p. 7) and Hankel (loc. cit., p. 3) state that the term "associative" goes back to Hamilton but give no reference. The word is used for the first time by Hamilton in Proceedings of the Royal Irish Academy, vol. II, p. 430.

¶ Transactions R. I. A., vol. XXI, p. 227.

** "Lectures on Quaternions," preface, p. 31.

†† Life of Hamilton, vol. III, p. 247.

when, and only when,

$$a_i = b_i, \qquad (i = 1, \ldots, n)$$

de Morgan* first gives for $n = 3$.

De Morgan, in 1844,† discusses the general commutative and associative systems with a modulus in three units. He arrives at several particular systems which he considers distinct though in somec ases they are equivalent, and gives several examples of divisors of zero. His systems, as well as those considered by Hamilton, have real parameters.

The pluquaternions of Kirkman,‡ the clefs of Cauchy,§ Grassmann's "Ausdehnungslehre," Hankel's|| alternating numbers, and the Situation Kalcul of Scheffler make little or no advance in the problem of classifying number systems, though Grassmann admits complex parameters. Important investigations were made by Frobenius, C. S. Peirce, and Clifford, subsequent to the appearance of Peirce's memoir, and on that account are not noticed here.

Peirce's Problem.

We have seen that previous to 1870 the number in n units and operations on it were defined, and the conditions imposed on the constants of multiplication that a closed system be associative were stated. There was, however, no method suggested by which one closed system could be grouped with an infinite number of other systems and a classification of all closed systems arrived at. The great usefulness of certain systems as Cauchy's clefs and quaternions would indicate that by an exhaustive search after all number systems, others might be found which would be adapted to various domains of research. The problem that Peirce set for himself was:**

1°. To enumerate exhaustively all types of number systems for a small number of units.

2°. To develop a calculus for some or all of these systems.

3°. To make application of these calculuses.

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* Transactions of the Cambridge Philosophical Society. vol. VII, p. 177, 1842.

† Trans. Cam. Philos. Soc., vol. VIII, p. 241, 1844.

‡ Philosophical Magazine, 3rd Series, vol. XXXIII, pp. 447 and 494, 1848.

|| Comptes Rendus, vol. XXXVI; pp. 70 and 129, 1853.

§ Loc. cit., p. 119.

** Loc. cit., pp. 98 and 119.

The first part of the problem occupies practically all of Peirce's published work on the subject.

In order to discuss the two questions proposed at the beginning of the paper, I shall treat in turn the principles of classification used by Peirce, and show their relation to those employed by more recent investigators. These principles of classification are five in number.

I. *Number* of *Units.*—Systems in the same number of units are thrown into the same group. This principle Peirce uses in common with all other writers on the subject.

II. *Equivalence.*—Two systems in the same number of units, $e_1, e_2 \ldots . e$, and $e'_1, e'_2 \ldots . e'_n$, are considered equivalent if linear relations exist of the type

$$e'_k = \sum_{i=1}^{n} a_{ki} e_i, \qquad k = 1 \ldots . n.$$

Peirce evidently assumes, though not explicitly, that $|a_{ki}| \neq 0$, and that for any value of k, the coefficient $a_{kk} \neq 0$. The latter unnecessary restriction is not always followed by the editor, Mr. C. S. Peirce.* This devise of classification Peirce justly considers one of the leading elements of originality in his work This principle has been used by most subsequent writers.†

III. *Pure Systems.*—Peirce further distinguishes between pure and mixed systems, but proposes to enumerate only the former. His definitions of these terms are not precise, but an inspection of their use shows that a system is called mixed if its units may be divided into two or more groups (which may have common units), such that the product, if non-vanishing, of two units of the same group is a number in that group, while the product of units which are not found in the same group is zero.‡ For illustration, on p. 123 [13], we have the system

	i	j	k
i	i	j	0
j	0	0	0
k	k	0	0

* Loc. cit., for example, p. 136 (foot-note); p. 205 (foot-note).

† Equivalent systems are of the same "*typus*" (see Study, Göttinger Nachrichten, 1889, p. 240).

‡ Cayley gives substantially this definition for the case $n = 2$, Collected Works, XII, p. 61.

In this mixed system the groups are i, j and i, k. Evidently the product inside each group is in that group while the products of j and k vanish. On p. 129 [12], we have

	i	j	k	l
i	i	j	k	l
j	j	k	0	0
k	k	0	0	0
l	0	0	0	0

In this mixed system the groups are i, j, k and i, l.

Again, on p. 133 [162] we have

	i	j	k	l
i	i	j	0	0
j	0	0	0	j
k	k	0	0	0
l	0	0	k	l

The groups here are i, j; i, k; l, k, and l, j.

Peirce's definitions of pure and mixed algebras correspond exactly to the definitions of irreducible and reducible number systems used by Scheffers,* except that the groups in a reducible system can contain no common units. Thus in Scheffer's enumeration appear several systems that Peirce rejects as not pure. In fact, every system given by Study or Scheffers which is not included in Peirce's enumeration he rejects as mixed, either in particular or under some general rule. An understanding of the definitions under which Peirce placed himself renders some of the criticisms of his work less forcible. For example, Study† states that Peirce gives only one of the five existant distinct systems in

* Mathematische Annalen, vol. 39, p. 317.

† Göttinger Nachrichten, 1889, p. 240.

three units. On comparing the systems given by the two writers, we see that all but one, (III), of Study's* systems are mixed and are rejected by Peirce on that ground. This system Peirce gives.

IV. *Reciprocity.*—Study† has defined two systems to be reciprocals of each other when the multiplication table of one is derived from that of the other by an interchange of rows and columns. Peirce gives no definition corresponding to this, but his usage follows it rather closely, and he assumes throughout his memoir that reciprocal systems are "virtual repetitions" of each other; for example, on p. 121, Peirce derives the system in two units

	i	j
i	i	j
j	0	0

(1)

and states that

	i	j
i	i	0
j	j	0

(2)

is a "virtual repetition." Notwithstanding the fact that Peirce assumes continually that such systems are not to be distinguished from each other, Cayley‡ complains that (2) is lacking since (1) and (2) are not equivalent. Bucheim§ in attempting to show what Peirce means by "virtual repetition" and to justify his position, arrives at a necessary and sufficient condition that two systems be reciprocals. In only one respect Peirce does fail to follow the present usage in regard to reciprocal systems. If a subsystem in one system is the reciprocal of a subsystem of another, while in other respects the systems are identical, Peirce calls them "virtual repetitions" of each other. For example, on p. 154 [111] the following systems are not considered distinct, since the subsystems formed by the units l, m in the systems are reciprocal.

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* Loc. cit., p. 246.

† Loc. cit., p. 240.

‡ Collected Works, XII, p. 61.

§ Messenger of Mathematics, vol. XV, 1886.

	i	j	k	l	m
i	i	j	0	0	0
j	0	0	$a_{23}i$	$b_{24}j$	$b_{25}j$
k	k	$d_{32}l + e_{32}m$	0	0	0
l	0	0	$c_{43}k$	l	m
m	0	0	$c_{53}k$	0	0

	i	j	k	l	m
i	i	j	0	0	0
j	0	0	$a_{23}i$	$b_{24}j$	$b_{25}j$
k	k	$d_{32}l + e_{32}m$	0	0	0
l	0	0	$c_{43}k$	l	0
m	0	0	$c_{53}k$	m	0

V. *Idempotent Numbers.*—Peirce defines an idempotent number as one, say α, for which a positive integer m exists so that $\alpha^m = \alpha$. If we define the modulus of a system as the number μ such that $x\mu = \mu x = x$ for every number x in the system, we see that the existence of an idempotent number is the necessary, though not sufficient, condition to the existence of a modulus. Peirce divides his systems into two groups, one of which contains systems with an idempotent, while the other contains systems without such a number. The former systems he considers of chief importance, and derives other systems only to assist in obtaining systems with an idempotent number. Thus he obtains all pure, inequivalent systems with a modulus, and several with no modulus: for example, each of the systems f_4, e_4, p. 130; h_4, p. 133, possesses an idempotent number, but no modulus.

The very close relation that exists between systems with idempotent units and those with a modulus, and the modifications necessary to make in Peirce's method, to derive all such systems are indicated elsewhere by the writer.*

We can now state precisely the problem that Peirce set for himself. He aimed to develop so much of the theory of hypercomplex numbers as would

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* Loc. cit.

enable him to enumerate all inequivalent, pure, non reciprocal number systems in less than seven units. The relation to the problem treated by Scheffers is plain, if we remember that the first two of Peirce's principles of classification are identical with those of Scheffers, and the other three are only slightly modified. Peirce solved this problem completely. The theorems stated by him are in every case true, though in some cases his proofs are invalid.

It is remarkable that Peirce did not avail himself of the clear and compact definitions of equality, and the fundamental operations given by De Morgan and Hamilton. Had this course been followed it must have lead to a lucidity of development the lack of which Study and Molien justly deplore.

YALE UNIVERSITY, *February*, 1901.

Groups Defined by the Orders of Two Generators and the Order of their Product.

By G. A. Miller.

If the three numbers l, m, n are the orders of three operators such that one of these operators is the product of the other two, it is known that the group generated by any two of these operators is completely defined by l, m, n, provided two of these numbers are equal to 2; or one is 2, the other 3 while the third is one of the three numbers 3, 4, 5.* In what follows we shall prove that these two operators may be so selected as to generate any one of an infinite system or groups of finite orders whenever l, m, n do not satisfy one of the given conditions. In particular, if only the orders of two operators and that of their product are fixed, and if at least two of these orders exceed three, these operators can be so chosen as to generate any one of an infinite system of groups of a finite order. Hence two such operators generate a group of infinite order whenever they satisfy no conditions except such as depend upon their orders and the order of their product.

The method employed may be briefly described as follows: Since the order of a transitive substitution group of degree n is a multiple of n and a divisor of $n!$ this order is finite or infinite as n is finite or infinite. If two substitutions of fixed orders can be so chosen that they generate a transitive substitution group whose degree is any multiple of a certain number, the order of such a group may have any one of an infinite number of values. Hence these substitutions can be so chosen as to generate any one of an infinite system of groups. The object is, therefore, to find two substitutions of the required orders such that their product is of the required order, and that they generate a transitive group whose degree is an arbitrary multiple of some number.

In what follows the three operators will be represented by L, M, N and

* Bulletin of the American Mathematical Society, vol. VII, 1901, p. 424.

their orders by l, m, n respectively. It will be assumed that $ML = N$ and that $l \leqq m \leqq n$, since these assumptions do not affect the generality of the results. The cases when $l = 2$ and when $l = 3$ will be considered separately, beginning with the former. The product of the following two substitutions

$$M_1 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot a_7a_8a_9 \cdot \cdots \cdot a_{6k-2}a_{6k-1}a_{6k},$$
$$L_1 = a_3a_4 \cdot a_5a_7 \cdot a_6a_8 \cdot a_9a_{10} \cdot a_{11}a_{13} \cdot a_{12}a_{14} \cdot \cdots \cdot a_{6k-3}a_{6k-2}$$

is composed of k cycles of degree 6, where k is any positive integer. In general, the product of the two substitutions

$$M_2 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot a_7a_8a_9 \cdot \cdots \cdot a_{3ak-2}a_{3ak-1}a_{3ak},$$
$$L_2 = a_3a_4 \cdot a_6a_7 \cdot \cdots \cdot a_{3\alpha-3}a_{3\alpha-2} \cdot a_{3\alpha-1}a_{3\alpha+1} \cdot a_{3\alpha}a_{3\alpha+2} \cdot \cdots \cdot a_{3ak-3}a_{3ak-2}$$

is composed of k cycles of degree 3α, $\alpha > 1$. Hence two substitutions of orders two and three respectively, whose product is an arbitrary multiple of three, can be so chosen as to generate any one of an infinite system of groups of finite order. When $\alpha > 3$, there are at least two letters which do not occur in L_2, in each of the cycles of degree 3α mentioned above. By adding transpositions to L_2 each of these cycles can clearly be made of degree $3\alpha + 1$ or of degree $3\alpha + 2$ instead of 3α. When $\alpha = 3$, we may make the cycles of degree 10 by the same method. The remaining cases, i. e. when $n = 7, 8$ or 11 may be treated separately as follows, the substitutions being chosen in such a manner as to give the fundamental transitive group by omitting the last cycle of M and the last two cycles of L. To obtain the generators of the transitive group whose degree is any multiple of the degree of this fundamental transitive group, it is only necessary to increase the subscripts of L and M by multiples of this degree.

$$M_3 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot a_7a_8a_9 \cdot a_{10}a_{11}a_{12} \cdot a_{13}a_{14}a_{15} \cdot a_{16}a_{17}a_{18} \cdot a_{19}a_{20}a_{21} \cdot a_{22}a_{23}a_{24} \cdot a_{25}a_{26}a_{27} \cdot a_{28}a_{29}a_{30} \cdot a_{31}a_{32}a_{33} \cdot a_{34}a_{35}a_{36} \cdot a_{37}a_{38}a_{39} \cdot a_{40}a_{41}a_{42} \cdot a_{43}a_{44}a_{45},$$

$$L_3 = a_3a_4 \cdot a_5a_7 \cdot a_6a_{10} \cdot a_8a_{12} \cdot a_{11}a_{13} \cdot a_9a_{20} \cdot a_{14}a_{16} \cdot a_{15}a_{29} \cdot a_{18}a_{29} \cdot a_{21}a_{22} \cdot a_{23}a_{25} \cdot a_{24}a_{35} \cdot a_{26}a_{28} \cdot a_{30}a_{31} \cdot a_{32}a_{34} \cdot a_{33}a_{38} \cdot a_{36}a_{37} \cdot a_{39}a_{40} \cdot a_{41}a_{43} \cdot a_{42}a_{44},$$

$$M_3L_3 = a_1a_2a_4a_7a_{12}a_6a_3 \cdot a_5a_{10}a_{13}a_{16}a_{17}a_{19}a_9 \cdot a_8a_{20}a_{22}a_{25}a_{28}a_{15}a_{11} \cdot a_{14}a_{29}a_{31}a_{34}a_{24}a_{21}a_{18} \cdot a_{23}a_{35}a_{37}a_{33}a_{30}a_{26}a_{27} \cdot a_{32}a_{38}a_{40}a_{43}a_{42}a_{39}a_{36} \cdot a_{41}a_{44} \cdot \cdots,$$

$$M_4 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot a_7a_8a_9 \cdot a_{10}a_{11}a_{12} \cdot a_{13}a_{14}a_{15} \cdot a_{16}a_{17}a_{18} \cdot a_{19}a_{20}a_{21} \cdot a_{22}a_{23}a_{24} \cdot a_{25}a_{26}a_{27},$$

$$L_4 = a_3a_4 \cdot a_5a_7 \cdot a_6a_{21} \cdot a_8a_{19} \cdot a_9a_{18} \cdot a_{11}a_{15} \cdot a_{12}a_{17} \cdot a_{13}a_{22} \cdot a_{14}a_{16} \cdot a_{23}a_{25} \cdot a_{24}a_{26},$$

$$M_4L_4 = a_1a_2a_4a_7a_{19}a_{20}a_6a_3 \cdot a_5a_{21}a_8a_{18}a_{14}a_{11}a_{17}a_9 \cdot a_{10}a_{15}a_{22}a_{25}a_{24}a_{13}a_{16}a_{12} \cdot a_{23}a_{26} \cdot \cdots,$$

$$M_5 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot a_7a_8a_9 \cdot a_{10}a_{11}a_{12} \cdot a_{13}a_{14}a_{15},$$

$$L_5 = a_3a_4 \cdot a_5a_7 \cdot a_6a_{10} \cdot a_8a_9 \cdot a_{11}a_{13} \cdot a_{12}a_{14},$$

$$M_5L_5 = a_1a_2a_4a_7a_9a_5a_{10}a_{13}a_{12}a_6a_3 \cdot a_{11}a_{14} \cdot \cdots$$

This completes the proof that there is an infinite number of groups for every value of $n > 5$ whenever $l = 2$ and $m = 3$. We proceed to the cases when $l = 2$ and $m = 4$. It is easy to see that the product of the following two substitutions is composed of k cycles of degree 4:

$$M_6 = a_1a_2a_3a_4 \cdot a_5a_6a_7a_8 \cdots a_{4k-3}a_{4k-2}a_{4k-1}a_{4k},$$
$$L_6 = a_3a_5 \cdot a_4a_6 \cdot a_7a_9 \cdot a_8a_{10} \cdots a_{4k-5}a_{4k-3} \cdot a_{4k-4}a_{4k-2}.$$

From this it follows that we may select L in such a manner that LM is of order 4α, and that L, M generate a transitive group of degree $4\alpha k$, α and k being arbitrary. To effect this, it is only necessary to select L in such a manner as to connect some of the cycles of M by a single transposition, while others are connected in the same manner as in L_6. When $\alpha > 2$, the order of the cycles of N can be increased by three or less and diminished by at least one by means of additional transpositions in L. When $\alpha = 2$, the order of N can clearly be increased by one or two in the same manner. It remains only to consider the cases when n is one of the three members 5, 6, 7, The following substitutions prove the existence of an infinite number of groups in each of these cases:

$$N_7 = a_1a_2a_3a_4a_5 \cdot a_6a_7a_8a_9a_{10} \cdot a_{11}a_{12}a_{13}a_{14}a_{15},$$
$$L_7 = a_3a_6 \cdot a_4a_8 \cdot a_5a_7 \cdot a_9a_{11} \cdot a_{10}a_{12},$$
$$N_7L_7 = a_1a_2a_6a_5 \cdot a_3a_8a_{11}a_{10} \cdot a_4a_7 \cdot a_9a_{12} \cdots$$
$$M_8 = a_1a_2a_3a_4 \cdot a_5a_6a_7a_8 \cdot a_9a_{10}a_{11}a_{12},$$
$$L_8 = a_3a_6 \cdot a_4a_5 \cdot a_7a_9 \cdot a_8a_{10},$$
$$M_8L_8 = a_1a_2a_6a_9a_8a_4 \cdot a_3a_5 \cdot a_7a_{10} \cdots$$
$$M_9 = a_1a_2a_3a_4 \cdot a_5a_6a_7a_8 \cdot a_9a_{10}a_{11}a_{12} \cdot a_{13}a_{14}a_{15}a_{16},$$
$$L_9 = a_3a_{10} \cdot a_4a_6 \cdot a_5a_9 \cdot a_7a_\alpha \cdot a_8a_\beta \cdot a_{11}a_{13} \cdot a_{12}a_{14},^*$$
$$M_9L_9 = a_1a_2a_{10}a_{13}a_{12}a_5a_4 \cdot a_3a_6a_\alpha a_7a_\beta a_8a_9 \cdot a_{11}a_{14} \cdots$$

From the two substitutions M_6, L_6 it follows directly that it is possible to select a substitution of order $m > 3$ and a substitution of order 2 so that the order of their product is an arbitrary multiple of m, and that they generate a transitive group whose degree is an arbitrary multiple of the order of their product. When $m > 6$, we can evidently make the order of this product any number $\geqq m$ by the addition of transpositions in L. Moreover, it is clear that

* Since two new elements are added by L_9, the remaining subscripts are obtained by increasing the given ones by 12 instead of by 14.

$n = 9$ is the only case which requires special considerations when $m = 6$. That there is also an infinity of groups in this case follows from the facts that there are substitutions of orders two and six whose product is of order 9,* and that there is an infinite number of groups when $l = 2$, $m = 3$, $n = 9$. When $m = 5$, there is also only one special case, viz. $n = 7$. The following two substitutions prove the existence of an infinite number of groups of finite order in this case:

$$M_{10} = a_1a_2a_3a_4a_5 \cdot a_6a_7a_8a_9a_{10} \cdot a_{11}a_{12}a_{13}a_{14}a_{15} \cdot a_{16}a_{17}a_{18}a_{19}a_{20},$$

$$L_{10} = a_4a_{13} \cdot a_5a_{11} \cdot a_8a_9 \cdot a_{10}a_{12} \cdot a_{14}a_{16} \cdot a_{15}a_{17},$$

$$M_{10}L_{10} = a_1a_2a_3a_{13}a_{16}a_{15}a_5 \cdot a_4a_{11}a_{10}a_6a_7a_9a_{12} \cdot a_{14}a_{17} \ldots.$$

This completes the consideration of all the cases when $l = 2$. Some additional cases may be reduced, in the following manner, to those which have been considered above. If a group is generated by two operators (s_1, s_2), s_1 and all its transforms with respect to s_2 generate either the entire group or an invariant subgroup whose index under the entire group is a divisor of the order of s_2. In particular, if the order of s_2 is 2, s_1 and its transform with respect to s_2 generate either the entire group or a subgroup of half its order. In this special case, when $s_2^2 = 1$, the product of s_1 and $s_2^{-1}s_1s_2$ is $(s_1s_2)^2$. Hence, it follows from the preceding paragraphs that two operators of the same order > 2 whose product is also of a higher order than 2, can always be so selected that they generate any one of an infinite system of groups. For instance, from the fact that two operators of orders 2 and 5 respectively, whose product is of order 8, can be so selected as to generate any one of an infinite system of groups of finite order, it follows that two operators of order 5 whose product is of order 4 have the same property. Since the group generated by two given operators is identical with the one generated by one of these operators and the product of the operators, it remains only to consider the cases when no two of the numbers l, m, n have a common factor > 2 and when $l > 2$.

When $l = 3$ and $m = 4$, we need to consider only one value of n, viz. 5, since in all the other cases it has been proved that M can be replaced in part by an operator of order two, which can be so selected that any one of some infinite system of groups is generated by L and M. The following substitutions prove

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* American Journal of Mathematics, vol. XXII, 1900, p. 185.

that this special case does not present an exception to the general theorem under consideration:

$$M_{11} = a_1a_2a_3a_4 \,.\, a_5a_6a_7a_8 \,.\, a_9a_{10}a_{11}a_{12}, \quad L_{11} = a_4a_5a_6 \,.\, a_7a_9a_\alpha \,.\, a_8a_{11}a_\beta,$$
$$M_{11}L_{11} = a_1a_2a_3a_5a_4 \,.\, a_6a_9a_{10}a_\beta a_8 \,.\, a_7a_{11} \ldots . \,*$$

When $l = 3$ and $m = 5$, we may assume that n is odd, for if it were divisible by 4 or contained either 3 or 5 as a factor, it would come under the preceding cases. Hence it must contain some odd factor greater than 5. It is now clear that it may be assumed that both m and n are odd whenever $m > 4$. Since the product of the two substitutions $a_1a_2a_3a_4a_5 \,.\, a_6a_7a_8a_9a_{10} \,.\, a_{11}a_{12}a_{13}a_{14}a_{15} \ldots . \, a_l$ and $a_5a_6a_7 \,.\, a_{10}a_{11}a_{12} \ldots . \, a_la_1a_2$ is of order five and involves an arbitrary multiple of 5 letters, and since $a_1a_2a_3a_4a_5 \,.\, a_6a_7a_8a_9a_{10}$ into $a_5a_6a_\alpha$ is a cyclic substitution of degree 11, it is evident that when $l = 3$ and $m > 4$, L and M can be so chosen as to generate a transitive group whose degree is any multiple of a fixed number.†

It may now be assumed that each of the three numbers l, m, n is odd and exceeds 3. The method employed in the preceding paragraph requires only slight modification to apply to all of these cases. It is based upon the following two propositions: 1). In combining two cycles, any odd number ($< l - 1$) of letters may be added; e. g. $a_1a_2a_3a_4a_5a_6a_7 \,.\, a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}$ into $a_7a_8a_\alpha a_\beta a_\gamma$ and into $a_7a_8a_9a_{10}a_\alpha$ gives substitutions of order 17 and 15 respectively. 2). In closing a cycle, any even number ($< l - 2$) of letters may be added; e. g. $a_1a_2a_3a_4a_5a_6a_7 \,.\, a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}$ into $a_7a_8a_\alpha a_\beta a_9 \,.\, a_{14}a_1a_{\alpha_1}a_{\beta_1}a_2$ and into $a_7a_8a_{10}a_{11}a_9 \,.\, a_{14}a_7a_3a_4a_2$ gives substitutions of orders 9 and 7 respectively, each of which is composed of two cycles. These propositions combined with the fact that the order of N can be increased by any even number $< m$ by means of additional cycles in L complete the proof of the required theorem, viz. *If two of the three numbers l, m, n are equal to 2 or if $l = 2$, $m = 3$, and $n = 3$, 4, or 5, the group generated by L and M is completely defined by the values l, m, n. For all other sets of values of l, m, n any two of the substitutions L, M, N can be so constructed as to generate any one of some infinite system of groups of finite order.*

ITHACA, N. Y., *August*, 1901.

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* The fundamental transitive group is of degree 10 and the subscripts are increased by multiples of 8 to obtain the other transitive groups. To obtain a definite transitive group, the last cycle of L is to be changed so as to contain two letters not found in M.

† C. f. American Journal, l. c.

Canonical Form of a Linear Homogeneous Transformation in an Arbitrary Realm of Rationality.

By Leonard Eugene Dickson.

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1. In volume XXII of the *American Journal of Mathematics*, the writer investigated the canonical forms of linear homogeneous transformations in a finite field (necessarily a Galois Field). It was shown that the type of canonical form obtained by M. Jordan for the case of the field of integers taken modulo p is capable of immediate generalization to an arbitrary Galois Field. Instead of the direct, but lengthy, method of proof employed by M. Jordan, the paper cited gives a proof by induction from a smaller to a greater number of irreducible factors of the characteristic determinant. The latter method leads to a canonical form of linear transformations in an arbitrary field F. The formal process of reduction is the same as in the case of a Galois Field.* But the proof that the reduced form has the desired properties† is necessarily more complicated for the general field F than for a Galois Field. Indeed, the content of the theorem is essentially greater for the general field, since the roots of an equation $F(K) = 0$, belonging to and irreducible in F, may not all be rational functions in F of a single root, as is the case when F is a Galois Field (see §4). The object of this paper is to give the desired additional proof and thereby establish the validity of the canonical form for an arbitrary field.‡ In §6 is given an example to illustrate the generalized theorem.

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* *American Journal*, vol. XXII, pp. 121–127. The paper will be designated A. J.

† Properties (a) and (b) below (§3), corresponding to (a) and (b) of A. J., p. 123.

‡ The theorem is stated in a different, but more compact, form by the writer in the Transactions of the American Mathematical Society, vol. 2 (1901), p. 393.

2. Proceeding as in A. J., pp. 124–6, we obtain the preliminary form S_2:

$$\lambda_i' = K_i\lambda_i, \qquad (i = 0, 1, \ldots, k-1)$$

$$Y_{01}' = K_0 Y_{01} + \alpha_{10}\lambda_0, \qquad Y_{0j}' = K_0(Y_{0j} + Y_{0j-1}), \quad (j = 2, \ldots, a_1)$$

$$Y_{0a_1+1}' = K_0 Y_{0a_1+1} + \beta_{10}\lambda_0, \quad Y_{0a_1+j}' = K_0(Y_{0a_1+j} + Y_{0a_1+j-1}), \quad (j = 2, \ldots, a_2)$$

. . . . . .

$$Z_{01}' = L_0 Z_{01}, \qquad Z_{0j}' = L_0(Z_{0j} + Z_{0j-1}), \quad (j = 2, \ldots, b_1)$$

. . . . . .

the expressions for Y_{ij}', Z_{ij}' being conjugate to those for Y_{0j}', Z_{0j}'. We proceed to prove that α_{10}, β_{10} are polynomials in K_0 with coefficients in F, and that the variables Y_{ij}, Z_{ij} have the properties:

(a). The variables Y_{0j} are linear homogeneous functions of the initial variables $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in K_0 with coefficients in F. The Y_{ij} are derived from the Y_{0j} by replacing K_0 by K_i.

(b). The variables Z_{0j} are linear homogeneous functions of $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in L_0 with coefficients in F. The Z_{ij} are derived from the Z_{0j} by replacing L_0 by L_i.

By the hypothesis made for the induction, the variables η_{ij} have the properties (a), the variables ζ_{ij} have the properties (b). Also

$$\lambda_i = X_0 + K_i X_1 + K_i^2 X_2 + \ldots + K_i^{k-1} X_{k-1}, \tag{1}$$

where the X_j are linear homogeneous functions of $\xi_1, \ldots, \xi_m$ with coefficients in F. Hence, in the transformation $\overline{S}$ (A. J., bottom of p. 125), the sums

$$\sum_{i=0}^{k-1} \alpha_{ji}\lambda_i, \quad \sum_{i=0}^{k-1} \beta_{ji}\lambda_i$$

are linear functions of $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in K_0 with coefficients in F. Applying the lemma below (§3),

$$\alpha_{ji} = \phi_j(K_0, K_i), \quad \beta_{ji} = \psi_j(K_0, K_i), \tag{2}$$

where ϕ_j, ψ_j are rational functions of their arguments with coefficients in F. Similarly, $\sum_{i=0}^{k-1} \alpha'_{ji}\lambda_i$ is a linear function of $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in L_0 with coefficients in F. Hence,

$$\alpha'_{ji} = \chi_j(L_0, K_i), \tag{3}$$

where χ_j is a rational function with coefficients in F.

Now (A. J., p. 126) $Z_{0s} \equiv \zeta_{0s} + \sum_{i=0}^{k-1} B_{si}\lambda_i$, where

$$B_{1i} = \frac{\alpha'_{1i}}{L_0 - K_i}, \quad B_{2i} = \frac{\alpha'_{2i}}{L_0 - (K_i)} - \frac{L_0\,\alpha'_{1i}}{(L_0 - K_i)^2}, \ldots.$$

In view of (1) and (3), it follows that $\sum B_{1i}\lambda_i$, $\sum B_{2i}\lambda_i$, ... are symmetric in $K_0, K_1, \ldots, K_{k-1}$, and hence are rational functions of $\xi_1, \ldots, \xi_m$ and L_0 with coefficients in F. A like result therefore holds for $Z_{01}, Z_{02}, \ldots$. In the field F a rational function of L_0 equals a polynomial in L_0.

Next (A. J., p. 126), $Y_{0s} \equiv \eta_{0s} + \sum_{i=0}^{k-1} A_{si}\lambda_i$, where

$$A_{10} = \frac{\alpha_{20}}{K_0}, \quad A_{1i} = \frac{\alpha_{1i}}{K_0 - K_i}, \quad A_{20} = \frac{\alpha_{30}}{K_0}, \quad A_{2i} = \frac{\alpha_{2i}}{K_0 - K_i} - \frac{K_0\,\alpha_{1i}}{(K_0 - K_i)^2},$$

etc., for $i = 1, 2, \ldots, k-1$, while $A_{a_1 0}$, $A_{a_1 + a_2 0}$, ... remain undetermined. In view of (1) and (2), it follows that A_{10}, A_{20}, ... are rational functions of K_0 with coefficients in F, and that the sums

$$\sum_{i=1}^{k-1} A_{1i}\lambda_i, \quad \sum_{i=1}^{k-1} A_{2i}\lambda_i, \ldots.$$

are symmetric in $K_1, \ldots, K_{k-1}$, and hence are rational functions of $\xi_1, \ldots, \xi_m$ and K_0 with coefficients in F. A like result therefore holds for $Y_{01}, Y_{02}, \ldots, Y_{0 a_1 - 1}$. The only modification necessary for $Y_{0 a_1}$, $Y_{0 a_1 + a_2}$, ... is to *choose* $A_{a_1 0}, A_{a_1 + a_2 0}, \ldots$ to be rational functions of K_0 with coefficients in F. It follows that every Y_{0j} is a linear function of $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in K_0 with coefficients in F.

Incidentally, it was shown that $\alpha_{10}, \beta_{10}, \ldots$ are polynomials in K_0 with coefficients in F. Denote them by $\phi(K_0), \psi(K_0), \ldots$ Proceeding as in A. J., p. 127, we obtain the desired canonical form upon variables possessing the properties (a) and (b).

3. LEMMA.—*Let the λ_i be defined by relations* (1), *and let the γ_i be independent of $\xi_1, \ldots, \xi_m$. If, in the linear function*

$$\lambda \equiv \sum_{i=0}^{k-1} \gamma_i \lambda_i$$

of $\xi_1, \ldots, \xi_m$, the coefficients are polynomials in σ with coefficients in the field F, then will

$$\gamma_i = f(\sigma, K_i), \qquad (i = 0, 1, \ldots, k-1)$$

where f is a polynomial in σ and K_i with coefficients in F.

By hypothesis, the expression

$$\lambda \equiv X_0 \Sigma\gamma_i + X_1 \Sigma K_i\gamma_i + X_2 \Sigma K_i^2\gamma_i + \ldots + X_{k-1}\Sigma K_i^{k-1}\gamma_i$$

is a linear function of $\xi_1, \ldots, \xi_m$ whose coefficients are polynomials in σ with coefficients in F. Since $X_0, X_1, \ldots, X_{k-1}$ are linearly independent linear functions of $\xi_1, \ldots \xi_m$ with coefficients in F, it follows that

$$\Sigma\gamma_i = f_0, \quad \Sigma K_i\gamma_i = f_1, \quad \Sigma K_i^2\gamma_i = f_2, \ldots, \quad \Sigma K_i^{k-1}\gamma_i = f_{k-1},$$

where $f_0, f_1, \ldots, f_{k-1}$ are polynomials in σ with coefficients in F. Hence

$$\Delta_i\gamma_i = \begin{vmatrix} f_0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ f_1 & K_0 & K_1 & \ldots & K_{i-1} & K_{i+1} & \ldots & K_{k-1} \\ f_2 & K_0^2 & K_1^2 & \ldots & K_{i-1}^2 & K_{i+1}^2 & \ldots & K_{k-1}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ f_{k-1} & K_0^{k-1} & K_1^{k-1} & \ldots & K_{i-1}^{k-1} & K_{i+1}^{k-1} & \ldots & K_{k-1}^{k-1} \end{vmatrix} \equiv D_i,$$

where Δ_i is derived from D_i by replacing $f_0, f_1, f_2, \ldots, f_{k-1}$ by $1, K_i, K_i^2, \ldots, K_i^{k-1}$ respectively. Since Δ_i is a product of the differences of the K_j, it does not vanish. By the interchange of K_a with K_b, where a and b differ from i, Δ_i and D_i only change in sign. Hence γ_i is symmetric in $K_0, K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_{k-1}$. It follows that γ_i is a rational function of $f_0, f_1, \ldots, f_{k-1}$ and K_i with coefficients in F,

$$\gamma_i = \omega(f_0, f_1, \ldots, f_{k-1}; K_i),$$

the form of the function ω being the same for each value of i. Moreover, ω is linear in $f_0, f_1, \ldots, f_{k-1}$. Hence γ_i is a polynomial in σ whose coefficients are rational functions (and hence polynomials) in K_i with coefficients in F.

4. For a finite field, the proof may be somewhat simplified. In that case, $K_0, K_1, K_2, \ldots, K_{k-1}$ are polynomials in a single root K_i. In view of their origin, the Y_{0j} are linear functions of $\xi_1, \ldots, \xi_m$ whose coefficients are rational functions of $K_0, K_1, \ldots, K_{k-1}$ with coefficients in F, and, therefore, are polynomials in K_0 with coefficients in F, if F be a finite field. But the Z_{0j} might involve $K_0, K_1, \ldots, K_{k-1}$ as well as L_0. By means of the above lemma, it was readily shown that the Z_{0j} involve only L_0. An independent proof* is given in §5. The method involved does not seem to apply to the case of the variables Y_{0j} for a general field F.

5. In view of the hypothesis for the induction, we may set

$$\eta_{ij} \equiv y_{0j} + K_i y_{1j} + K_i^2 y_{2j} + \ldots + K_i^{k-1} y_{k-1\,j}, \qquad (i = 0, 1, \ldots, k-1)$$

where the y_{ij} are linear homogeneous functions of $\xi_1, \ldots, \xi_m$ with coefficients in F. The y_{ij} are linearly independent since the η_{ij} are. Since the determinant $|K_i^j| \neq 0$, $y_{0j}, y_{1j}, \ldots, y_{k-1\,j}$ can be expressed as linear functions of $\eta_{0j}, \eta_{1j}, \ldots, \eta_{k-1\,j}$. Similarly, $\zeta_{0j}, \zeta_{1j}, \ldots, \zeta_{k-1\,j}$ depend upon independent functions $z_{0j}, z_{1j}, \ldots, z_{k-1\,j}$ belonging to F; while the latter depend upon the former. Set

$$\zeta_{ij} = f_i(z_{0j}, \ldots z_{k-1\,j}), \quad z_{ij} = f_i'(\zeta_{0j}, \ldots, \zeta_{k-1\,j}). \qquad (i = 0, 1, \ldots, k-1).$$

In view of formulæ (1), $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ depend upon $X_0, X_1, \ldots, X_{k-1}$, and inversely.

Take as independent variables the functions X, y, z. Then $\overline{S}$ replaces each X by a function of the X only, each y by a function of the X and the y only, each z by a function of the X and the z only, the coefficients of these functions being in F. Let $\overline{S}$ replace z_{ij} by $\sum_{r,s} m_{rs} z_{rs} + \sum_r n_r X_r$. It is possible to

* It is merely a new form of that by Jordan, "Traité," pp. 121–122.

determine a set of functions

$$\bar{z}_{ij} \equiv z_{ij} + \sum_r x_{ijr} X_r, \tag{4}$$

such that $\bar{S}$ replaces $\bar{z}_{ij}$ by $\sum_{r,s} m_{rs} \bar{z}_{rs}$. One set is given by

$$\bar{\bar{z}}_{ij} \equiv f_i'(Z_{0j}, \ldots, Z_{k-1j}),$$

where the Z_{is} are the functions determined in A. J., p. 126. To show that this is the only set, consider any set (4). In terms of the variables λ_i, η_{ij}, $\bar{\zeta}_{ij} \equiv f_i(\bar{z}_{0j}, \ldots, \bar{z}_{k-1j})$, the transformation $\bar{S}$ replaces each by ζ_{ij} by a linear function of the $\bar{\zeta}_{is}$ only. Now $\bar{\zeta}_{ij}$ has the form $\zeta_{ij} + \sum_r \gamma_{ijr} X_r$. But the only set of functions of this form which $\bar{S}$ replaces by functions in which the coefficients of the X_r vanish was shown to be the set Z_{ij}. Hence $\bar{\zeta}_{ij} \equiv Z_{ij}$, so that $\bar{\bar{z}}_{ij} \equiv f_i'(\bar{\zeta}_{0j}, \ldots, \bar{\zeta}_{k-1j}) = \bar{\bar{z}}_{ij}$. But, when $\bar{S}$ is expressed in terms of the variables K, y, z, the unique set (4) can be determined by the method of undetermined coefficients, so that the x_{ir} will belong to the field F. Hence the functions $\bar{z}_{ij}$ belong to F. But the coefficients in $f_i(z_{0j}, \ldots . z_{k-1j})$ are polynomials in L_i with coefficients in F. Hence the same is true for

$$f_i(\bar{z}_{0j}, \ldots, \bar{z}_{k-1j}) \equiv \bar{\zeta}_{ij} = Z_{ij}.$$

6. EXAMPLE.—The general theory is well illustrated by the following example: Consider a linear transformation whose characteristic determinant is $(\rho^3 - 2)^2$. By a general theorem (*Amer. Journ.*, vol. XXIII, p. 37), one such transformation is

$$A: \xi_1' = 4\xi_3 - 4\xi_6,\ \xi_2' = \xi_1,\ \xi_3' = \xi_2,\ \xi_4' = \xi_3,\ \xi_5' = \xi_4,\ \xi_6' = \xi_5.$$

The functions which A multiplies by ρ, where $\rho^3 = 2$, are

$$\xi_1 + \rho\xi_2 + \rho^2\xi_3 - 2\xi_4 - 2\rho\xi_5 - 2\rho^2\xi_6.$$

Denote the function by λ_1, λ_2 or λ_3 according as $\rho = \sqrt[3]{2}$, $\rho = \omega\sqrt[3]{2}$, or $\rho = \omega^2\sqrt[3]{2}$ respectively, where $\omega \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ is an imaginary cube root of

unity. Taking as new variables the linearly independent functions $\lambda_1, \lambda_2, \lambda_3, \xi_4, \xi_5, \xi_6$, the transformation A becomes

$$\begin{cases} \lambda_1' = \sqrt[3]{2}\,\lambda_1, \quad \lambda_2' = \omega\sqrt[3]{2}\,\lambda_2, \quad \lambda_3' = \omega^2\sqrt[3]{2}\,\lambda_3, \\ \xi_4' = 2\xi_6 + f, \quad \xi_5' = \xi_4, \quad \xi_6' = \xi_5, \end{cases} \qquad [f \equiv \tfrac{1}{6}\sqrt[3]{2}\,(\lambda_1 + \omega\lambda_2 + \omega^2\lambda_3)].$$

The following transformation of determinant 2

$$\xi_4' = 2\xi_6, \quad \xi_5' = \xi_4, \quad \xi_6' = \xi_5,$$

multiplies the function $\xi_4 + \rho\xi_5 + \rho^2\xi_6\,(\rho^3 = 2)$ by ρ. Set

$$\begin{aligned} y_1 &= \xi_4 + \sqrt[3]{2}\,\xi_5 + \sqrt[3]{4}\,\xi_6, \\ y_2 &= \xi_4 + \omega\sqrt[3]{2}\,\xi_5 + \omega^2\sqrt[3]{4}\,\xi_6, \\ y_3 &= \xi_4 + \omega^2\sqrt[3]{2}\,\xi_5 + \omega\sqrt[3]{4}\,\xi_6. \end{aligned}$$

Then A replaces y_1, y_2, y_3 by, respectively,

$$\sqrt[3]{2}\,y_1 + f, \quad \omega\sqrt[3]{2}\,y_2 + f, \quad \omega^2\sqrt[3]{2}\,y_3 + f.$$

By introducing as new variables

$$\begin{aligned} x_1 &= \tfrac{1}{6}\lambda_1, \quad \eta_1 = y_1 + ax_1 + (\omega x_2 - x_3)/(1-\omega), \\ x_2 &= \tfrac{1}{6}\lambda_2, \quad \eta_2 = y_2 + bx_2 + (\omega x_3 - x_1)/(1-\omega), \\ x_3 &= \tfrac{1}{6}\lambda_3, \quad \eta_3 = y_3 + cx_3 + (\omega x_1 - x_2)/(1-\omega), \end{aligned}$$

we obtain the canonical form

$$\begin{aligned} &x_1' = \sqrt[3]{2}\,x_1, \qquad x_2' = \omega\sqrt[3]{2}\,x_2, \qquad x_3' = \omega^2\sqrt[3]{2}\,x_3, \\ &\eta_1' = \sqrt[3]{2}\,(\eta_1 + x_1), \quad \eta_2' = \omega\sqrt[3]{2}\,(\eta_2 + x_2), \quad \eta_3' = \omega^2\sqrt[3]{2}\,(\eta_3 + x_3). \end{aligned}$$

Here x_1, x_2, x_3 are of the respective forms

$$\theta(\xi_1, \ldots, \xi_6;\ \sqrt[3]{2}), \quad \theta(\xi_1, \ldots, \xi_6;\ \omega\sqrt[3]{2}), \quad \theta(\xi_1, \ldots, \xi_6,\ \omega^2\sqrt[3]{2}), \quad (5)$$

where θ is a linear function of its arguments, the coefficients being rational

numbers. Again, we have

$$\eta_1 = \xi_4 + \sqrt{2}\,\xi_5 + \sqrt[3]{4}\,\xi_6 + \frac{a}{6}\lambda_1 + \tfrac{1}{6}(-\xi_1 + \sqrt[3]{4}\,\xi_3 + 2\xi_4 - 2\sqrt[3]{4}\,\xi_6),$$

$$\eta_2 = \xi_4 + \omega\sqrt[3]{2}\,\xi_5 + \omega^2\sqrt[3]{4}\,\xi_6 + \frac{b}{6}\lambda_2 + \tfrac{1}{6}(-\xi_1 + \omega^2\sqrt[3]{4}\,\xi_3 + 2\xi_4 - 2\omega^2\sqrt[3]{4}\,\xi_6),$$

$$\eta_3 = \xi_4 + \omega^2\sqrt[3]{2}\,\xi_5 + \omega\sqrt[3]{4}\,\xi_6 + \frac{c}{6}\lambda_3 + \tfrac{1}{6}(-\xi_1 + \omega\sqrt[3]{4}\,\xi_3 + 2\xi_4 - 2\omega\sqrt[3]{4}\,\xi_6).$$

Hence η_1, η_2, η_3 will be of the respective forms (5), if we take

$$a = f(\sqrt[3]{2}), \quad b = f(\omega\sqrt[3]{2}), \quad c = f(\omega^2\sqrt{2}),$$

where f is a rational function with coefficients in F. Whatever be the field F the variables in the canonical form have the desired properties (a).*

THE UNIVERSITY OF CHICAGO, *October*, 1901.

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*I take this occasion to note certain errata in my article in vol. XXIII:

Page 351, line 25, *for* $\lambda\xi_1$ *read* $\gamma\xi_1$.
Page 364, line 20, *for* 2^m *read* $2m$.
Page 366, bottom, *for*), *read*).
Page 373, bottom, *for* F^λ *read* F_λ.

A New Theory of Collineations and their Lie Groups.

By H. B. Newson.

Introduction.

1. Among the groups of transformations discovered by Lie and used by him to illustrate his theory of continuous groups, the most important by far are the projective group and its subgroups. But a projective transformation, or collineation, may be studied synthetically as well as analytically, and hence arise the possibility and the desirability of a purely geometric theory of collineations and their groups. This consideration has led the writer to seek to develop such a theory. The investigation has been undertaken in the spirit of the following from Klein's "Erlangen Program":

"Es ist zwar immer an der Forderung festzuhalten, dass man einen mathematischen Gegenstand noch nicht als erledigt betrachten soll, so lange er nicht begreiflich evident geworden ist, und es ist das Vordringen an der Hand des Formalismus eben nur ein erster aber schon sehr wichtiger Schritt."

Consequently, the present memoir is in no sense to be regarded as a contribution to Lie's theory of continuous groups, but its object is rather to set forth an independent geometric theory of the projective group of the plane. Starting with Lie's concepts of Transformation and Group, I have developed a new theory which is not based upon the infinitesimal transformation. A similar theory has already been published* by the writer for the projective transformations of the points on a line and the extension of the theory to the projective group in three dimensions is an easy and natural step.

Certain theorems and results reached in the above-mentioned paper will recur here, and reference will usually be made to that paper. Two important

* "Continuous Groups of Projective Transformations Treated Synthetically." Part I. Kansas University Quarterly, vol. IV, pp. 71–93.

theorems on projective transformations in one dimension not found in the above paper will be frequently used and will be proved when introduced.

§1.—PERSPECTIVE COLLINEATIONS; THEIR PROPERTIES AND LIE GROUPS.*

2. *Construction of a Perspective Collineation of the Plane.* Let p and p' be two planes intersecting in a line l and making any convenient angle with one another. From O, a point not in either plane, draw a bundle of rays each intersecting p and p'. Any ray a of the bundle cuts p and p' in the points P and P', a pair of corresponding points in the two planes. In this way the field of points on p is projected into a new field of points on p'. If, now, p' be revolved about l until it coincides with p, the two fields of points, i. e. the original field and its projection on p', may be considered as existing in the same plane.

The result of these two operations of projection and revolution is to shift the points of the plane p into new positions so that there is a one-to-one correspondence between the old and new positions of the points. Evidently, the right lines of p are also shifted into new positions and remain right lines after the operation. Any operation which transforms points into points and right lines into right lines is called a collineation, or projective transformation, of the plane. The most simple form of a plane collineation is the one constructed above, and is called a perspective collineation. We shall use S to denote a perspective collineation.

3. *Invariant Figure of S.* Any point or line of p which, after the operation S, coincides with its corresponding point or line, is said to be invariant under S. Evidently, from the construction, every point on l is an invariant point of S. The perpendicular from O on the plane bisecting the angle plp' cuts p in A and p' in A'; the revolution about l brings A' to A, so that A is also an invariant point of S.

Let B be any point on l; the lines g and g' joining A and B, A' and B respectively are corresponding lines in the two planes. The revolution about l makes g' coincide with g, so that g is an invariant line of S. In like manner every line through A is an invariant line of S. Every invariant line has on it two invariant points, viz. A and the point where it crosses l.

*Many of the properties of perspective transformations contained in §1, were given in Professor A. Emch's dissertation, "On the Continuous Groups of Perspective Collineations in the Plane," published in the Kansas University Quarterly, Vol. V, pp. 1-36.

Every pencil of rays whose vertex is on l is invariant under S. Every invariant pencil of this sort has in it two invariant rays, viz. l and the ray through A. It should be noted that the invariant figure of S is a self-dualistic figure. The point A and the line l are called the vertex and axis respectively of the collineation S.

THEOREM 1.—*A perspective collineation S leaves invariant a self-dualistic figure consisting of a point A, every point on a line l, every line through A, and every pencil of rays with its vertex on l.*

4. *Characteristic Cross-Ratio of S.* Let q be the plane determined by O and l, and let r be the plane through l parallel to OA; let k be the cross-ratio of the pencil of planes $l\,(qrpp')$. Every line of the projecting bundle through O is cut by these four planes in ranges of four points, all of which have the same cross-ratio k. Any selected ray a of the bundle cuts these four planes in the points O, Q, P and P' respectively, and the cross-ratio $a\,(OQPP')$ is equal to k. If, now, every ray of the bundle be projected orthogonally on the plane p by lines parallel to OA, the four points O, Q, P, P' will be projected into A, B, P, P_1 and the cross-ratio $(ABPP_1) = k$. In like manner the cross-ratio of the two invariant points and every pair of corresponding points on every line through A is constant and equal to k.

Thus the perspective collineation S produces a one-dimensional projective transformation along every line of the plane through A, and the characteristic cross-ratio has the same value along all these lines. In like manner S produces a one-dimensional projective transformation in every pencil of lines with its vertex on l, and evidently the characteristic cross-ratio has the same value k in all these pencils.

THEOREM 2.—*A perspective collineation S is completely characterized by its invariant figure and a cross-ratio k which is the same along all invariant lines (except l) and through all invariant points (except A).*

5. *One-Parameter Group of Perspective Collineations.* The perspective collineation S is not the only one that can be constructed which leaves invariant the figure (A, l). Take O_1, a second point on OA, and proceed as before to construct a second collineation S_1. The invariant figures of S and S_1 will be the same, but the cross-ratios k and k_1 will not be the same; for the reason that the

cross-ratios of the axial pencils $l(qrpp')$ and $l(q_1rpp')$, where q_1 is the plane determined by l and O_1, are not the same. Thus we see that an unlimited number of perspective collineations can be constructed, leaving invariant the figure (A, l).

Consider, now, the operation whose effect on the points of the plane is equivalent to the effect of the successive operations of S and S_1. Since S and S_1 both transform points into points and lines into lines, their *resultant* must be a collineation; and, since S and S_1 both leave the figure (A, l) invariant, their resultant must also leave (A, l) invariant, and hence is a perspective collineation.

Let S transform the point P into P_1; the points P and P_1 are collinear with A and their joining line cuts l at some point B. Let S_1 transform the point P_1 to P_2; P_2 will also lie on the line AB. The resultant of S and S_1 is S_2, a perspective collineation which transforms the point P directly to P_2. This may be expressed by the equation $SS_1 = S_2$.

The characteristic cross-ratios of S, S_1 and S_2 are respectively $k = (ABPP_1)$, $k_1 = (ABP_1P_2)$ and $k_2 = (ABPP_2)$. Expanding these equations,

$$k = \frac{AP}{BP} : \frac{AP_1}{BP_1}, \quad k_1 = \frac{AP_1}{BP_1} : \frac{AP_2}{BP_2}, \quad k_2 = \frac{AP}{BP} : \frac{AP_2}{BP_2}.$$

Eliminating the fraction $\frac{AP_1}{BP_1}$ from the first and second by multiplication, we have

$$kk_1 = \frac{AP}{BP} : \frac{AP_2}{PP_2} = k_2.$$

Thus we see that the cross-ratio of S_2 is equal to the product of the cross-ratios of S and S_1.

We have now found that there are an unlimited number of perspective collineations leaving the figure (A, l) invariant, and that the resultant of any two of these is another of the same system. This is what is known as the *group property*, and hence we conclude that the aggregate of all perspective collineations leaving (A, l) invariant is a group. The natural parameter of this Lie group is the characteristic cross-ratio k.

THEOREM 3.—*The aggregate of all perspective collineations having the same vertex and axis is a Lie group of one parameter; the natural parameter of the group*

is the cross-ratio k; the cross-ratio of the resultant of any two collineations of the group is equal to the product of the cross-ratios of the components, i. e. $k_2 = kk_1$.

6. *Properties of the Group $H_1(A, l)$.* For the sake of generality, we shall suppose that the point A and the line l are either real or imaginary, in general, imaginary. The points of the plane as P, P_1, etc., are also either real or imaginary, and hence the cross-ratio k is, in general, a complex number. Evidently there is a collineation in the group, which will be designated by $H_1(A, l)$, for every value of k.

The most important properties of the group may be enumerated as follows: If $k=1$ in the cross-ratio $k=(ABPP_1)$, P_1 must coincide with P, since A and B are separate points. Thus every point in the plane is an invariant point under this collineation, which is the identical collineation of the group.

Two collineations of the group for which k and k_1 have reciprocal values, have the effect of exactly neutralizing each other. Their resultant is the identical collineation, and they are called inverse collineations. Every collineation in the group has an inverse also in the same group.

There are two collineations in the group which are their own inverses, viz. those for which $k=1$ and $k=-1$; the first is the identical collineation and the second is called the involutoric collineation of the group. The effect of the involutoric collineation is to interchange every pair of corresponding points in the plane.

All collineations in the group for which $k=1+d$, where d is an infinitesimal complex number, are called infinitesimal collineations of the group. Since d may be written in the form $\delta e^{i\theta}$, where θ varies from 0 to 2π, we see that the group contains an infinite number of infinitesimal collineations. If the values of k are represented in the usual manner by points in the complex plane, the infinitesimal collineations are represented by points infinitely near to the unity point and are distinguished according to the directions of their points from the unit point.

Let the infinitesimal collineation $k=1+\delta e^{i\theta}$ be repeated n times; the resultant is a collineation of the group for which $k=(1+\delta e^{i\theta})^n$, a number which can be made to differ as much as we please from 1 by taking n sufficiently large. If θ be constant, the locus of the point k, as n varies, is a logarithmic spiral about the origin. There is a separate spiral for each infinitesimal collineation of the

group. Every collineation, whose cross-ratio is represented by a point on one of these spirals, is said to be generated from the infinitesimal collineation of the spiral. These spirals all pass through the unit point and lie infinitely close together. There are two families of these spirals, one right-handed and the other left-handed. They cover twice over the entire plane, and hence every collineation in the group, except those for which $k = 0$ and $k = \infty$, may be generated from at least two infinitesimal collineations; those for which k is real and positive may be generated from three infinitesimal collineations.*

The two collineations for which $k = 0$ and $k = \infty$ are essentially different in character from all other collineations in the group. Their corresponding points in the plane of k are the points toward which all the spirals converge without ever reaching them. The effect of the collineation $k = 0$ is to transform every point of the plane, except A, to a point on l; the effect of the other one, $k = \infty$, is to transform every point of the plane, except those on l, to A. These two *pseudo*-transformations are called the essentially singular collineations of the group.

A Lie group is defined to be continuous when every transformation in it can be generated from an infinitesimal transformation of the group. Hence the group of perspective collineations $H_1(A, l)$ is continuous everywhere except for $k = 0$ and $k = \infty$.†

THEOREM 4.—*The one-parameter group of perspective collineations* $H_1(A, l)$ *is continuous everywhere in the plane of its complex parameter* k, *except at* $k = 0$ *and* $k = \infty$.

7. *Resultant of Two Coaxial Perspective Collineations.* Let S and S_1 be two perspective collineations having the same axis l and their vertices at A and A_1 respectively. Let the line joining A and A_1 be l', cutting the axis l at B; let k and k_1 be the cross-ratios of S and S_1 respectively. The resultant of S and S_1 is a collineation and leaves invariant every point on the axis l. The line l' is an invariant line of both S and S_1, and is, therefore, also an invariant line of S_2.

* For a fuller discussion of this theory, see the Bulletin of the American Mathematical Society (2), vol. IV, pp. 113–114.

† In his definition of a projective transformation, Lie explicitly lays down the condition that the determinant of the transformation shall not vanish; this condition excludes from consideration the transformations of the group for which $k = 0$ and ∞. Lie's group thus defined is everywhere continuous for the reason that he has excluded the discontinuity points.

Along the line l' we have two one-dimensional projective transformations whose invariant points are (A, B) and (A_1, B) respectively. Their resultant is a one-dimensional projective transformation whose invariant points are A_2 and B, where A_2 is some other point on l'. Since S_2 leaves invariant every point on l and the point A_2, it must be a perspective collineation, and also must leave invariant every line through A_2 and every pencil with vertex on l.

Since S_2 is a perspective collineation, it is characterized by a constant cross-ratio k_2, which we wish to determine. Consider the pencil of lines through the point B; the invariant rays of this pencil in all three collineations S, S_1 and S_2 are l and l'. The one-dimensional transformations of this pencil due to S and S_1 have cross-ratios k and k_1; hence, the cross-ratio of their resultant is $k_2 = kk_1$. Hence, the cross-ratio of one of the invariant pencils of S_2 is $k_2 = kk_1$; but it has been proved that a perspective collineation is characterized by a constant cross-ratio in all invariant pencils and along all invariant lines. Therefore, the characteristic cross-ratio of S_2 is everywhere equal to the product of the cross-ratios of S and S_1.

THEOREM 5.—*The resultant of two coaxial perspective collineations is a perspective collineation having the same axis; the characteristic cross-ratio of the resultant is equal to the product of the cross-ratios of the components.*

8. *Resultant of Two One-Dimensional Transformations with a Common Invariant Point.* From the last result we are able to infer an important theorem for one-dimensional projective transformations. Along the line l' the two one-dimensional transformations have one invariant point B in common, and it was shown above that the cross-ratio of the resultant is equal to the product of the cross-ratios of the components. In every pencil with vertex on l, except at B, the two one-dimensional projective transformations have only one invariant ray l in common and the cross-ratio of the resultant is equal to the product of the cross-ratios of the components. We know also that this law of combination of the cross-ratios always holds good when the two invariant elements are in common. We are able, therefore, to state the following theorem:*

* This theorem properly belongs to the subject of one-dimensional projective transformations, but I had not observed it at the time the former paper was written. An analytic proof was given by the writer in Bulletin American Mathematical Society (2), vol. IV, p. 110. It was there shown that the

THEOREM 6.—*The cross-ratio of the resultant of two one-dimensional projective transformations, which have one or both invariant elements in common, is equal to the product of the cross-ratios of the components.*

9. *Invariant Figure and Properties of an Elation.* The vertex A_2 of the resultant of two coaxial perspective collineations may be anywhere on the line l'. We wish to examine the case when A_2 coincides with B. The cross-ratio of the resultant is given by $(A_2BPP_2) = kk_1$. When $kk_1 = k_2 = 1$, either $P = P_2$ or $A_2 = B$. In the first case, the resultant is the identical collineation; in the second case, it is a special kind of perspective collineation in which the vertex A falls on the axis l. This special kind of perspective collineation is called an Elation. It is the resultant of two coaxial perspective collineations having different vertices and reciprocal values of k.

In the case of an elation, there is no longer a characteristic cross-ratio, for there are only three elements, e. g. B, P and P_1; but a new relation can be found between the segments BP and BP_1. We shall suppose, for simplicity, that the line l' is perpendicular to l. When A and B are distinct points, we have $k = (ABPP_1)$, whence $(APBP_1) = 1 - k$. Expanding, we get

$$\frac{AB}{PB} \cdot \frac{PP_1}{AP_1} = 1 - k \quad \text{or} \quad \frac{PP_1}{PB \,.\, AP_1} = \frac{1-k}{AB}.$$

Writing $AP_1 - AP$ for PP_1, we get

$$\frac{AP_1 - AP}{PB \,.\, AP_1} = \frac{1-k}{AB}.$$

Taking the limit of both sides of the last equation as A approaches B and setting $\lim_{A=B} \frac{1-k}{AB} = \alpha$, we find the relation

$$-\frac{1}{BP} + \frac{1}{BP_1} = \alpha \quad \text{or} \quad \frac{1}{BP_1} = \frac{1}{BP} + \alpha,$$

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cross-ratio k_2 of the resultant of any two one-dimensional transformations is given by

$$\frac{k_2 + 1}{\sqrt{k_2}} = \frac{kk_1 - k - k_1 + 1}{\sqrt{kk_1}} \lambda + \frac{k + k_1}{\sqrt{kk_1}},$$

where $\lambda = (A_1ABB_1)$, one of the cross-ratios of the two pairs of invariant points (AB) and (A_1B_1). When $A = A_1$ or $B = B_1$, or both are simultaneously equal, $\lambda = 1$ and $k_2 = kk_1$.

where α is a constant for all pairs of corresponding points, because k is such a constant. Let P_i be the point on l' into which the point at infinity is transformed. Then

$$\frac{1}{BP_i} = \frac{1}{\infty} + \alpha;$$

thus α is the reciprocal of the segment BP_i.

The elation S' leaves invariant every point on the axis l; and every invariant pencil with vertex on l has only one invariant ray, viz. l. Let us consider one of these pencils with vertex at C on l. We have in the general case $C(ABPP_1) = k$, whence $C(APBP_1) = 1 - k$. Expanding this,

$$\frac{\sin(ACB)}{\sin(PCB)} \cdot \frac{\sin(PCP_1)}{\sin(ACP_1)} = 1 - k \quad \text{or} \quad \frac{\sin(PCP_1)}{\sin(PCB)\,.\sin(ACP_1)} = \frac{1-k}{\sin(ACB)}.$$

For PCP_1 write $BCP_1 - BCP$, whence,

$$\frac{\sin(BCP_1)\cos(BCP) - \cos(BCP_1)\sin(BCP)}{\sin(PCB)\,.\sin(ACP_1)} = \frac{1-k}{\sin(ACB)}.$$

Take the limit of both sides as A approaches B and put

$$\lim_{A=B} \frac{1-k}{\sin(ACB)} = \lim_{A=B} \frac{AC(1-k)}{AB} = \alpha\,.BC.$$

We thus find $-\cot(BCP) + \cot(BCP_1) = BC\alpha$; let $BCP = \theta$, $BCP_1 = \theta_1$, and $BC = a$; we then have

$$\cot\theta_1 = \cot\theta + a\alpha.$$

This relation is constant for all pairs of corresponding rays in any given pencil whose vertex is at a distance a from B.

When the vertex of the pencil is at B, $a = 0$; in this case $\theta_1 = \theta$, and every ray of this pencil is an invariant ray of S'. This is as it should be, for in a perspective collineation S, of which S' is a special case, every ray through the vertex is invariant.

Along every ray of the pencil through B we have a one-dimensional projective transformation of the second* kind t', viz. with only one invariant point B.

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* There are two kinds of one-dimensional projective transformations; the first kind has two invariant elements, and is designated by t, the second kind has one invariant element, and is designated by t'.

Along l'', any ray of this pencil making an angle ϕ with l, the characteristic constant is not the same as along l'. Let g be the line through P parallel to l; S' transforms g into g_1, the line through P_1 parallel to l. Let g and g_1 cut l' in Q and Q_1; then Q is transformed by S' into Q_1. Since $AP = AQ \sin\phi$ and $AP_1 = AQ_1 \sin\phi$, we have the relation

$$\frac{1}{AQ_1} = \frac{1}{AQ} + \alpha \sin\phi.$$

An elation S' is thus seen to have properties quite different from those of a perspective collineation S. The invariant figure of S' consists of all points on l and all lines through A, some point of l. S' produces along all lines through A, and in all pencils with vertices on l, one-dimensional projective transformations of the second kind, viz. with only one invariant element. It is fully characterized by its invariant figure and a constant α.

THEOREM 7.—*The resultant of two coaxial perspective collineations with different vertices and reciprocal cross-ratios is an elation, having the same axis and its vertex collinear with the vertices of the components. An elation is completely determined by its invariant figure and a characteristic constant* α.

10. *The Group* $H_1'(Al)$ *and its Properties.* The aggregate of all elations with the same invariant figure Al has the group property and forms a group as we proceed to show. Let S' and S_1' be two such elations whose constants are α and α_1 respectively. The effect of S' on a point P of the plane is to move it to P_1 such that $\frac{1}{AP_1} = \frac{1}{AP} + \alpha \sin\theta$, where θ is the angle made by the line AP and l. The effect of S_1' on P_1 is to move it to P_2 such that $\frac{1}{AP_2} = \frac{1}{AP_1} + \alpha_2 \sin\theta$. Eliminating AP_1 from these equations, we find $\frac{1}{AP_2} = \frac{1}{AP} + (\alpha + \alpha_1)\sin\theta$; hence, the resultant of S' and S_1' is an elation with the same invariant figure and a characteristic constant $\alpha_2 = \alpha + \alpha_1$. Therefore, the aggregate of all elations with invariant figure Al form a group $H_1'(Al)$.

The group $H_1'(Al)$ contains an elation corresponding to every value of α, which is, in general, a complex number. Let the values of α be represented by points in the complex plane. The following properties of the group may be enunciated:

The group $H_1'(Al)$ contains one identical elation for which $\alpha = 0$. The elations of the group for which α and α_1 have equal numerical values but of opposite signs, are inverse to one another, and their resultant is the identical elation of the group. Every elation in the group has its inverse in the same group. The group contains ∞^1 infinitesimal elations represented by the points adjacent to the origin; the corresponding values of α are $\delta e^{i\theta}$. Let one of these infinitesimal elations be repeated n times; its final resultant is an elation of the group for which $\alpha = n\delta e^{i\theta}$. If θ is a constant, the locus of the point α as n varies is a half ray proceeding from the origin and making an angle θ with the axis of real numbers. Every elation of the group which is represented by a point on this half ray is generated from this infinitesimal elation. Each infinitesimal elation of the group has its corresponding half ray, and these rays lie infinitesimally close together and cover the entire plane. Thus every elation in the group (except that for which $\alpha = \infty$) is generated from an infinitesimal elation of the group. Hence the group is continuous everywhere in the plane of its parameter α except at the infinity point. The elation corresponing to $\alpha = \infty$ is an essentially singular elation of the group; its effect is to transform every point in the plane, except those on the line l, to the vertex A.

THEOREM 8.— *The one-parameter group of elations is continuous for all values of the parameter α except for $\alpha = \infty$.*

11. *Number of Perspective Collineations and Elations.* The number of perspective collineations in the plane is determined by the number of one-parameter groups of the kind $H_1(A, l)$ and $H_1'(Al)$. There are ∞^2 lines and ∞^2 points in the plane, and hence there are ∞^4 combinations of line and point such as (A, l) and ∞^3 lineal elements (Al) in the plane. Each of these figures is the invariant figure of a one-parameter group of perspective collineations; in the case that the point A falls on l, the resulting group is a group of elations. Hence, there are in the plane ∞^4 one-parameter groups of perspective collineations, ∞^3 of which are groups of elations. Since each one-parameter group contains ∞^1 collineations, there are ∞^5 perspective collineations in the plane, ∞^4 of which are elations.

We shall next concern ourselves with the question whether these ∞^5 collineations form a five-parameter group, and what groups of order higher than the

first may be built up out of these ∞^4 one-parameter groups. We shall first look for other groups of elations.

12. *The Group $H_2'(l)$.* Let us consider the aggregate of all coaxial one-parameter groups of elations, l being the axis. Every point on l is the vertex of such a group, and hence there are ∞^2 elations leaving invariant every point on l. To ascertain if this system of elations has the group property, we find the resultant of any two elations S' and S_1' of the system. Let S' be taken from the group whose vertex is at A and S_1' from the group with vertex at A_1; let their characteristic constants be α and α_1 respectively.

Let us first examine the effect of S' and S_1' on the pencil of lines through C, a chosen point on l. S' produces in the pencil (C) a one-dimensional projective transformation of the second kind of which l is the single invariant ray. If p and p_1 are two corresponding rays of the pencil making angles θ and θ_1 with l, we have

$$\cot\theta_1 = \cot\theta + a\alpha.$$

Let S_1' transform p_1 to p_2, where the angle between p_2 and l is θ_2; then we have the relation

$$\cot\theta_2 = \cot\theta_1 + a_1\alpha_1.$$

Eliminating θ_1 from these equations by addition, we have

$$\cot\theta_2 = \cot\theta + a_2\alpha_2,$$

where $a_2\alpha_2 = a\alpha + a_1\alpha_1$. This shows that S_2', the resultant of S' and S_1', is an elation whose axis is l and whose vertex is some point on l; for it produces a one-dimensional projective transformation of the second kind in every pencil with vertex on l. Thus the group property is established and we have found a two-parameter group of elations, the parameters being a and α; this group will be designated by $H_2'(l)$.

Let us next consider the effect of S' and S_1' on the pencil whose vertex is at infinity on l. Since the rays of the pencil are parallel to l, S' transforms g into g_1 such that $1/p_1 = 1/p + \alpha$, where p and p_1 represent the perpendiculars from l on the lines g and g_1. S_1' transforms g_1 into g_2 so that $1/p_2 = 1/p_1 + \alpha_1$; eliminating p_1, we get $1/p_2 = 1/p + \alpha_2$, where $\alpha_2 = \alpha + \alpha_1$. Thus the values of a_2 and α_2 are obtainable from the equations

$$a_2\alpha_2 = a\alpha + a_1\alpha_1 \quad \text{and} \quad \alpha_2 = \alpha + \alpha_1.$$

Since the values of a_2 and α_2 are unaltered when a and a_1, α and α_1 are interchanged, it follows that the elations of the group $H_2'(l)$ are commutative.

THEOREM 9.—*The doubly infinite system of elations having a common axis l forms a two-parameter group $H_2'(l)$ in which the elations are commutative.*

13. *The Group $H_2'(A)$.* We next consider the aggregate of all one-parameter groups of elations which have a common vertex A and whose axes are in turn every line through A. We shall find that this system also possesses the group property.

Since every elation of the system leaves invariant the point A and every line through A, it follows that the resultant of any two elations of the system must also leave this point and pencil invariant. Take any two elations of the system S' and S_1' whose axes are the lines l and l_1 and whose constants are α and α_1 respectively. Along every invariant ray of the pencil the resultant of the two one-dimensional projective transformations of the second kind with common invariant point at A, is also one of the second kind having the same invariant point A. Hence S_2', the resultant of S' and S_1', is also an elation with vertex at A and whose axis is some line l_2 through A. The group property is thus established.

In order to determine the value of α_2 and the position of the axis l_2 of S_2', we proceed as follows: Along any chosen ray g of the pencil through A, the constants of S', S_1' and S_2' are $\alpha \sin \phi$, $\alpha_1 \sin \phi_1$ and $\alpha_2 \sin \phi_2$, where ϕ, ϕ_1 and ϕ_2 are the angles which g makes with l, l_1 and l_2 respectively. But the constant of the resultant is equal to the sum of the constants of the components; hence, $\alpha_2 \sin \phi_2 = \alpha \sin \phi + \alpha_1 \sin \phi_1$.

Along l_2 the resultant is an identical transformation and its constant is 0; hence, we have $\alpha \sin (\phi_2 - \phi) + \alpha_1 \sin (\phi_2 - \phi_1) = 0$. Expanding and collecting, we get

$$(\alpha \cos \phi + \alpha_1 \cos \phi_1) \sin \phi_2 = (\alpha \sin \phi + \alpha_1 \sin \phi_1) \cos \phi_2 = \alpha_2 \sin \phi_2 \cos \phi_2, \quad \text{(a)}$$

whence, $\alpha \cos \phi + \alpha_1 \cos \phi_1 = \alpha_2 \cos \phi_2$.

From the two equations

$$\alpha_2 \sin \phi_2 = \alpha \sin \phi + \alpha_1 \sin \phi_1, \quad \text{and} \quad \alpha_2 \cos \phi_2 = \alpha \cos \phi + \alpha_1 \cos \phi_1, \quad \text{(b)}$$

we can calculate the values of a_2 and ϕ_2. Thus:

$$a_2 = \sqrt{a^2 + a_1^2 + 2aa_1 \cos(\phi - \phi_1)}. \qquad (c)$$

When $\phi_1 = \phi$, this reduces to $a_2 = a + a_1$, which is as it should be.

Evidently, from equations (b), the elations of the group $H_2'(A)$ are commutative.

THEOREM 10.—*The doubly infinite system of elations having a common vertex forms a two-parameter group $H_2'(A)$ in which the elations are commutative.*

14. *No other Groups of Elations.* There are only two other relative positions besides those discussed above in which two lineal elements Al and A_1l_1 can be placed:

1st. Let S' and S_1' be two elations whose lineal elements Al and A_1l_1 are so situated that A_1 lies on l. The effect of S' is to leave invariant A_1 and l, but no other line through A_1. If, now, S_1' operates on the points of the plane, it leaves invariant A_1 and l, but no other point on l. Hence, the resultant of S' and S_1' leaves invariant A_1 and l but no other points on l and no other lines through A_1. Therefore, the resultant S_2', although a collineation, is not an elation; and the aggregate of all elations whose invariant figures have this relative position, does not form a group. The resulting collineation in this case is very important, and will be discussed later. See article 62.

2d. In the second case, let S' and S_1' be two elations whose invariant lineal elements Al and A_1l_1 are in the most general relative position. Let l and l_1 intersect in A_2, and let the join of A and A_1 be l_2. The resultant of S' and S_1' leaves invariant A_2 and l_2, but not every point on l_2. Hence, the resultant S, though a collineation, is neither an elation nor a perspective collineation. Therefore, the aggregate of all the ∞^4 elations of the plane does not form a group.

Thus we see that there are only three kinds of groups of elations, viz. $H_1'(Al)$, $H_2'(l)$ and $H_2'(A)$.

15. *The Groups $H_3(l)$ and $H_2(lBl')$.* We found in article 7 that the resultant of S and S_1, two coaxial perspective collineations, is, in general, a perspective collineation S_2 whose axis is also l and whose vertex A_2 is collinear with A and A_1; and, furthermore, that $k = kk_1$. From these results we may draw important conclusions: 1st, the aggregate of all coaxial perspective collineations has the group property and forms a Lie group of three parameters, the param-

eters being k and the two coordinates of the vertex A; 2d, the aggregate of all coaxial perspective collineations whose vertices are collinear, forms a two-parameter Lie group, the parameters being k and one coordinate of A.

The first group, designated by $H_3(l)$, contains ∞^2 one-parameter subgroups $H_1(A, l)$, one for each point in the plane. It also contains ∞^2 subgroups of the second variety, designated by $H_2(lBl')$, one for each line l' in the plane. But two of these latter groups always contain one one-parameter subgroup in common, viz. $H_1(A_2, l)$, where A_2 is the intersection of l' and l_1'.

The group $H_2(lBl')$ contains ∞^1 subgroups $H_1(A, l)$, one for each point on l'; it also contains one subgroup of elations, viz. $H_1'(Bl)$; the group $H_3(l)$ contains one two-parameter subgroup of elations, viz. $H_2'(l)$.

Within the groups $H_3(l)$ and $H_2(lBl')$ the collineations are, in general, not commutative; the value of k_2 is independent of the order of S and S_1, but the position of A_2 is not independent of the order of S and S_1.

THEOREM 11.— *The aggregate of all coaxial perspective collineations forms a three-parameter group $H_3(l)$; the aggregate of all coaxial perspective collineations whose vertices are collinear forms a two-parameter group $H_2(lBl')$. The collineations of these groups are not commutative.*

16. *The Groups $H_3(A)$ and $H_2(Al'B)$.* We next proceed to find the resultant of two perspective collineations S and S_1 which have a common vertex A and whose axes l and l_1 intersect in B. It is evident that the resultant S_2 leaves invariant A, B, and every line through A. Also the two one-dimensional projective transformations in the pencil through B have one invariant l' in common, and hence their resultant leaves invariant l' and some second ray l_2 through B. Since A, l' and every line through A are invariant, every point on l_2 must also be invariant under S_2; and hence S_2 is also a perspective collineation with vertex at A. Thus the group property is established, and we infer that all perspective collineations with vertices at A form a three-parameter group $H_3(A)$. Along AB and also through B we have $k_2 = kk_1$, and hence the cross-ratio of S_2 is everywhere given by $k_2 = kk_1$.

From the above discussion we also infer that all perspective collineations whose vertices coincide and whose axes are concurrent in B form a two-parameter group, $H_2(Al'B)$. Evidently the group $H_2(Al'B)$ contains one subgroup of elations, $H_1'(Al')$; and $H_3(A)$ contains $H_2'(A)$ as a subgroup. The collineations of these groups are not commutative.

THEOREM 12.—*The aggregate of all perspective collineations which have a common vertex forms a three-parameter group, $H_3(A)$; within the group $H_3(A)$ all collineations whose axes are concurrent form a two-parameter subgroup, $H_2(Al'B)$. The collineations in these groups are not commutative.*

17. *No other Groups of Perspective Collineations.* We now proceed to show that the resultant of two perspective collineations, whose invariant figures are in any relative positions different from those discussed above, is not a perspective collineation, and hence there are no other varieties of groups except the eight already found. There are four other cases to be considered where the relative position of the invariant figures of S and S_1 are different from those above.

1st. Let the invariant figures (A, l) and (A_1, l_1) be so situated that A falls on l_1 and A_1 on l; let the intersection of l and l_1 be B_2 and the join of A and A_1 be l_2. The resultant of S and S_1 leaves invariant the triangle (AA_1B_2), but not every point on any side of it; hence, the resultant is not a perspective collineation, and all perspective collineations in this relative position do not form a group.

2d. Let the point A fall on l_1 but A_1 not on l, or vice versa; along l_2, the join of A and A_1, the two one-dimensional projective transformations result in another of the same kind, leaving invariant A and C, some other point on l_2. Thus the resultant leaves invariant a triangle (ACB_2) and is, in general, not a perspective collineation.

3d. Let the points A and A_1 of the first case be collinear with B_2; the resultant collineation leaves invariant B_2, A_2 some point on AA_1 and some line l_2 through B_2. But every line through A_2 will not be invariant, nor every point on l_2. Hence, the resultant is, in general, not a perspective collineation.

4th. Let the figures (A, l) and (A_1, l_1) be in the most general relative position so that l and l_1 meet in B_2, the join of A and A_1 does not pass through B_2 and A and A_1 do not fall on l_1 and l respectively. The resultant collineation leaves invariant a triangle $(A_2B_2C_2)$, where A_2 and C_2 are points on l_2, but it is not a perspective collineation. Hence, the aggregate of all ∞^5 perspective collineations of the plane do not form a five-parameter group.

Therefore, there are no four- or five-parameter groups of perspective collineations, and no other kinds of one-, two- and three-parameter groups besides those discussed above.

18. *Dualistic and Self-Dualistic Groups.* From the dualistic character of a collineation, we infer that its fundamental invariant figure is self-dualistic. When there exists a group of collineations, leaving a certain figure invariant, there must also exist a second group leaving invariant a figure dualistic to the first. Two such groups are said to be dualistic to one another; if the invariant figure of the group is a self-dualistic figure, the group is called a self-dualistic group.

As examples, we may cite the following pairs of dualistic groups: $H_3(A)$ and $H_3(l)$, $H_2(lBl')$ and $H_2(Al'B)$, $H_2'(A)$ and $H_2'(l)$; the one-parameter groups $H_1(A, l)$ and $H_1'(Al)$ are self-dualistic groups.

This principle holds for all collineations of whatever kind, and hereafter, when the existence of a certain group has been proved, it will be assumed, without further proof, that the dualistic group also exists and has properties dualistic to the first.

19. *Resumé of Perspective Collineations.* In the following list, the structure of all groups of perspective collineations is indicated. A dash above a letter indicates that the line or point thus marked takes on different positions in the invariant figure of the group. Thus $H_2'(l) = \infty\, H_1'(\bar{A}l)$ indicates that the point A takes all positions on the line l. Dualistic groups are bracketed together, and self-dualistic groups are bracketed alone.

$$
\begin{aligned}
&\{H_1'(Al)\},\\
&\{H_1(A, l)\},\\
&\left\{\begin{aligned} H_2'(l) &= \infty^1 H_1'(\bar{A}l)\\ H_2'(A) &= \infty^1 H_1'(A\bar{l})\end{aligned}\right.\\
&\left\{\begin{aligned} H_2(lBl') &= \infty^1 H_1(\bar{A}, l) + H_1'(Bl'),\\ H_2(ABl') &= \infty^1 H_1(A, \bar{l}) + H_1'(Al'),\end{aligned}\right.\\
&\left\{\begin{aligned} H_3(l) &= \infty^2 H_1(\bar{A}, l) + H_2'(l),\\ H_3(A) &= \infty^2 H_1(A, \bar{l}) + H_2'(A).\end{aligned}\right.
\end{aligned}
$$

§2.—Collineations with Invariant Triangle. Type I.

A.—*The Group $G_2(ABC)$ and its Subgroups.*

20. *The Two-Parameter Group $G_2(ABC)$.* The most general form of a collineation in the plane leaves invariant a triangle. Such a collineation is the

resultant of two perspective collineations whose invariant figures are so situated that the vertex of each is on the axis of the other. Thus, let S and S_1 be two such perspective collineations whose invariant figures are (A, l'') and (B, l') and whose cross-ratios are k and k_1 respectively. Let l'' and l' intersect in C, and let the join of A and B be l. S produces along the lines l and l' one-dimensional projective transformations whose cross-ratios are each k reckoned from A to B and C. (By this we mean that, if P and P_1 are two corresponding points on l, the cross-ratio k is that of the points taken in the order $(ABPP_1)$). S_1 produces along l and l'' one-dimensional projective transformations whose cross-ratios are each equal to k_1 reckoned from B to C and A. Thus along l we have the resultant of two one-dimensional transformations whose cross-ratios reckoned from A to B are k and $1/k_1$; hence, the cross-ratio of their resultant is k/k_1. The three cross-ratios taken all in the same order around the invariant triangle (ABC) are k/k_1, k_1 and $1/k$ and their product is unity.

Thus the resultant of two perspective collineations in this relative position is a non-perspective collineation which we shall designate by T. The fundamental properties of T are as follows: Its invariant figure is a triangle (ABC); along each side of the invariant triangle there is a one-dimensional transformation, and the product of the cross-ratios of these three transformations is unity when they are reckoned in the same order around the triangle; there is also a one-dimensional transformation of the pencils of rays through each vertex of (ABC), and the cross-ratios of these are the same as along the opposite sides. A collineation T is completely determined by its invariant triangle and its three cross-ratios.

Since the product of these three cross-ratios is unity, only two of them are independent. These two independent parameters may assume all possible values, and hence we see that there are ∞^2 different collineations of this type having the same invariant triangle (ABC). The resultant of any two of these collineations is evidently another of the same set, and hence the set forms a two-parameter group, which will be designated by $G_2(ABC)$.

THEOREM 13.—*There are ∞^2 collineations T, leaving one and the same triangle invariant; these form a two-parameter group $G_2(ABC)$; each collineation of this group is determined by the cross-ratios along the sides of the invariant triangle.*

21. *One-Parameter Subgroups of $G_2(ABC)$.* We shall now show that the two-parameter group $G_2(ABC)$ contains ∞^1 one-parameter subgroups. Let the

three cross-ratios along the sides AB, BC, and CA be k, k_1 and k_2; replace k_1 by k^{-r} and k_2 by k^{r-1}. The condition that the product of these cross-ratios around the triangle is unity is evidently fulfilled. A collineation T is now characterized by two constants k and r. Take from the group $G_2(ABC)$ two collineations $T(k, r)$ and $T_1(k_1, r_1)$ and form their resultant $T_2(k_2, r_2)$. Along the side AB we have $k_2 = kk_1$; along BC we have $k_2^{-r_2} = k^{-r} k_1^{-r_1}$; along CA we have $k_2^{r_2-1} = k^{r-1} k_1^{r_1-1}$.

If, now, T and T_1 be so chosen that $r_1 = r$, we see that also $r_2 = r$. Thus $k_2 = kk_1$, $k_2^{-r} = k^{-r} k_1^{-r}$ and $k_2^{r-1} = k^{r-1} k_1^{r-1}$. Hence, the resultant of two collineations in $G_2(ABC)$, for which the r's are the same, is a collineation with the same value of r. If r be kept constant and k made to vary, the resulting collineations have the group property and form a one-parameter subgroup of $G_2(ABC)$. There is a one-parameter subgroup of $G_2(ABC)$ for every value of r; thus the group $G_2(ABC)$ contains ∞^1 one-parameter subgroups, one of which is designated by $G_1(ABC)r$.

THEOREM 14.—*The two-parameter group $G_2(ABC)$ contains ∞^1 one-parameter subgroups $G_1(ABC)r$; each subgroup is characterized by a constant r, and its variable parameter is k.*

22. *The Path-Curves of $G_1(ABC)r$.* The analytic expressions for a one-parameter group $G_1(ABC)r$ can readily be written down. Let the triangle of reference be the invariant triangle (ABC), and let T be the collineation which transforms the point $P = (x, y, z)$ to $P_1 = (x_1, y_1, z_1)$. T is characterized by k and r; the cross-ratio of the pencil $C(ABPP_1)$ is k.

In terms of the coordinates of P and P_1 we have

$$\frac{z_1}{x_1} = k\frac{z}{x}\,;\quad \frac{x_1}{y_1} = k^{-r}\frac{x}{y}\,;\quad \frac{y_1}{z_1} = k^{r-1}\frac{y}{z}\,.$$

The effect of all the transformations of the one-parameter group $G_1(ABC)r$ upon a point P is to move it along a curve which is called its *path-curve*. There are ∞^1 of these path-curves so situated that every point of the plane lies on one, and only one, of these curves. The equation of this family of path-curves is found by eliminating k from the above equations; this gives

$$x^{r-1} yz^{-r} = x_1^{r-1} y_1 z_1^{-r} = C, \text{ or } x^{r-1} y = Cz^r.$$

Thus the path-curves of the group $G_1\,(ABC)\,r$ are curves of order r and form a family whose parameter is C.

THEOREM 15.—*When the invariant triangle is taken as the triangle of reference, the family of path-curves of a one-parameter group* $G_1\,(ABC)\,r$ *is given by* $x^{r-1}y = Cz^r$, *where* r *is a constant.*

23. *Geometric Meaning of* r. It is not difficult to determine, from the equation of the family of path-curves, the geometric meaning of r. Take any point $P = (x_1, y_1, z_1)$ and draw a tangent at P to the path-curve through P and join P to the vertices of the triangle (ABC). The equation of the tangent to the path-curve at P is

$$PT = x\,(r-1)\,y_1z_1 + yx_1z_1 - rzx_1y_1 = 0.$$

The lines PA, PB, and PC are given by the equations

$$PA = yx_1 - xy_1 = 0,$$
$$PB = yz_1 - zy_1 = 0,$$
$$PC = xz_1 - zx_1 = 0.$$

The cross-ratio of the pencil $P\,(BATC)$ is found to be r. It is evident that this cross-ratio r is a constant for every point on every path-curve of the group $G_1\,(ABC)\,r$.

THEOREM 16.—*The constant* r *in the one-parameter group* $G_1\,(ABC)\,r$ *is one of the cross-ratios of the pencil of four lines formed by a tangent to one of the path-curves and the three lines drawn from its point of contact to the vertices of the invariant triangle.*

Having the equation of the family of path-curves given in the form $x^{r-1}y = Cz^r$, their chief properties may readily be deduced. The curves will be algebraic or transcendental, according as r is a rational or irrational number. The vertices of the invariant triangle are singular points on all curves of the family, when r is irrational; one vertex is a singular point when r is rational, except when the path-curves are conics.

24. *Path-Curves when* $r = 1, 0, \infty$. The path-curves of a one-parameter subgroup of $G_2\,(ABC)$ reduce to straight lines for three values of r, viz. $r = (1, 0, \infty)$. Let $r = 1$ in $x^{r-1}y = Cz^r$, and we get a pencil of lines $y = Cz$ through B; let $r = 0$ and we get $y = Cx$, a pencil of lines through A; let $r = \infty$ and we get $x = Cz$, a pencil of lines through C.

The three cross-ratios around the sides of the invariant triangle are respectively k, k^{-r}, k^{r-1}, starting from AB. When $r = 1$, these become $k, k^{-1}, 1$; thus the one-dimensional transformation along CA is the identical transformation and every point on CA is an invariant point. Consequently, every line of the pencil through B is an invariant line. Hence, the subgroup of $G_2(ABC)$ for $r = 1$ is a group of perspective collineations. In like manner we see that when $r = 0$, we have a subgroup of perspective collineations with vertex at A and axis l''; when $r = \infty$, we have a third subgroup of perspective collineations.

Take any two of these perspective groups, as for example $H_1(A, l'')$ and $H_1(B, l')$; there are ∞^1 collineations in each group, and hence there are ∞^2 resultants obtained by combining each collineation of the one group with every one of the other. These ∞^2 resultants constitute the group $G_2(ABC)$; in fact, every collineation in $G_2(ABC)$ is the resultant of two perspective collineations, one from each of the groups $H_1(A, l'')$ and $H_1(B, l')$. Since there are three such subgroups of perspective collineations in $G_2(ABC)$, viz. $H_1(A, l'')$, $H_1(B, l')$ and $H_1(C, l)$, it follows that every collineation in $G_2(ABC)$ is in three different ways the resultant of two perspective collineations.

THEOREM 17.—*The group $G_2(ABC)$ contains three subgroups of perspective collineations, viz. the subgroups for which $r = 1, 0, \infty$. Every collineation in $G_2(ABC)$ is in three ways the resultant of two perspective collineations of the group.*

25. *Path-Curves when $r = -1, 2, 1/2$.* There are three other specially important subgroups of $G_2(ABC)$; these correspond to the values $r = -1, 2, 1/2$. Putting $r = -1$ in $x^{r-1}y = Cz^r$, we get $yz = Cx^2$; the path-curves reduce in this case to a system of conics having double contact at A and C. BA and BC are common tangents to the pencil of conics and AC is the chord of contact. In like manner, when $r = 2$, the path-curves are $xy = Cz^2$; when $r = 1/2$, the path-curves are $xz = Cy^2$. These are also pencils of conics having double contact; in each case two sides of the invariant triangle are common tangents and the third side is the chord of contact.

THEOREM 18.— *The group $G_2(ABC)$ contains three special subgroups for which the path-curves are conics having double contact; these are the subgroups for which $r = -1, 2, 1/2$.*

For any rational value of r, except the six special values $1, 0, \infty, -1, 2, 1/2$, there are six subgroups for which the path-curves are algebraic curves of the

same order, viz. for r, $1/r$, $1-r$, $1/(1-r)$, $(r-1)/r$, $r/(r-1)$. Thus for $r=3$ the path-curves are cubics, viz. $x^2y=Cz^3$; the path-curves are also cubics for $r=1/3$, -2, $-1/2$, $2/3$, $3/2$, and for no other values of r.

26. *Properties of the Group* $G_1(ABC)r$. Let the collineation T be a complex collineation, i. e. let the invariant points (ABC) be real or imaginary, and let the cross-ratios k, k_1, k_2 be complex numbers. The two parameters of the group $G_2(ABC)$ may be conveniently taken to be k and k', where k is the cross-ratio along AB and k' is that along AC. Then the cross-ratios of all collineations which belong to the subgroup $G_1(ABC)r$ satisfy the relation $k'=k^{1-r}$, the constant r is, in general, a complex number.

The independent parameter k of $G_1(ABC)r$ takes on all complex values, so that this group has many of the properties of the one-parameter group of perspective collineations. The identical collineation of the group $G_2(ABC)$ is given by $k=1$ and $k'=1$; this is contained in every one-parameter subgroup $G_1(ABC)r$. The group $G_1(ABC)r$ contains ∞^1 infinitesimal collineations from which the group may be generated. For $k=0$ and $k=\infty$ the group has two pseudo-collineations; it is continuous everywhere except at the points $k=0$ and $k=\infty$.

The subgroups of $G_2(ABC)$ for which r is rational (and hence real) have an important property which is not possessed by the groups for which r is irrational. To exhibit this property we proceed to find all collineations common to any two groups $G_1(ABC)r$ and $G_1(ABC)r'$. This is equivalent to finding all the roots common to the two equations $k'=k^{1-r}$ and $k'=k^{1-r'}$. Besides the identical and pseudo-collineations, these two groups have in common all collineations corresponding to values of k which satisfy the equation $k^{r-r'}=1$. If r is a rational number, $1-r$ is also rational; let even and odd integers be represented by e and o respectively, then we have three cases to be considered, viz. $1-r=e/o$, o/o, or o/e. In the first case, the equation $k'=k^{1-r}$ is satisfied by the pair of values $(-1, 1)$; in the second case, by the pair $(-1, -1)$; in the third case, by the pair $(1, -1)$. The collineation corresponding to the values $k=-1$ and $k'=1$ is the involutoric perspective collineation belonging to the group $H_1(B, l')$. This involutoric collineation belongs to all subgroups of $G_2(ABC)$ for which r is rational with odd numerator and odd denominator. In like manner, the collineation corresponding to the pair of values $(-1, -1)$ is the involutoric perspective collineation belonging to the group $H_1(A, l')$. This collineation

belongs to all subgroups of $G_2(ABC)$ for which r is rational with even numerator and odd denominator. The collineation corresponding to the pair of values $(1, -1)$ is the involutoric perspective collineation of the group $H_1(C, l)$; this belongs to all subgroups of $G_2(ABC)$ for which r is rational with odd numerator and even denominator. None of these three involutoric perspective collineations can belong to a subgroup of $G_2(ABC)$ for which r is irrational.

THEOREM 19.—*Every one-parameter subgroup of $G_2(ABC)$ for which r is a rational number contains an involutoric perspective collineation.*

27. *Number of Collineations of Type I.* We have seen that every collineation of type I leaves a triangle invariant and is further characterized by two independent cross-ratios k and k'. Every collineation of type I depends therefore upon eight parameters, viz. the six coordinates of the three vertices of the invariant triangle and these two independent cross-ratios. Since each of these parameters may assume ∞^1 different values, we see that there are ∞^8 collineations of type I in the plane. We are also enabled to distinguish two distinct kinds of variable parameters, viz. coordinates of invariant points and characteristic cross-ratios.

Since there are ∞^6 triangles in the plane, it follows that there are ∞^6 such two-parameter groups as $G_2(ABC)$ in the plane. Hence, the group of all collineations in the plane G_8 contains ∞^6 two-parameter subgroups $G_2(ABC)$. No two of these two-parameter groups can have a collineation of type I in common; for, if two collineations T and T_1 are identical, the eight parameters of the one must be equal to the eight parameters of the other. Now, the two cross-ratio parameters of the one may readily be equal to those of the other; but if the collineations leave different triangles invariant, all of the six coordinate parameters of the one cannot be equal to those of the other. Hence, T and T_1 cannot be identical.

28. *Three Classes of Groups of Type I.* Having now established the existence of two-parameter groups of the kind $G_2(ABC)$ and their one-parameter subgroups $G_1(ABC)r$, we shall proceed in the following paragraphs to enumerate the groups of higher orders which can be compounded out of these one- and two-parameter groups. In so doing, we shall find three distinct classes of such groups, viz. (1) those which are made up of two-parameter groups of the kind $G_2(ABC)$; (2) another class of groups in which every collineation is character-

ized by the same value of r; the groups of this class are made up of one-parameter groups of the kind $G_1(ABC)r$ in which r is a constant; (3) finally, a class of groups made up of one-parameter groups $G_1(ABC)r$, whose path-curves are conics, i. e. where $r = -1, 2, 1/2$. Strictly speaking, the third class is only a special case of the second class, but the special case is so important that these groups should be classified by themselves.

B.—Groups of the First Class.

By the aid of the principle that all collineations which leave a certain figure invariant, form a group, it is not difficult to enumerate all of the subgroups of the general projective group G_8 that can be compounded out of the ∞^6 two-parameter groups $G_2(ABC)$ in the plane.

29. *The Groups $G_6(l)$ and $G_6(A)$.* There are ∞^8 collineations of type I in the plane and only ∞^2 lines in the plane; hence, any line of the plane can be transformed into any other line or into itself in ∞^6 different ways. The ∞^6 collineations which transform a line l into itself, form a six-parameter group $G_6(l)$. There are ∞^4 triangles having the side l in common; each of these triangles is the invariant triangle of a two-parameter group $G_2(ABC)$. Thus we see that $G_6(l)$ is made up of ∞^4 two-parameter groups $G_2(ABC)$.

In like manner, we see that any point A of the plane is the invariant figure of a six-parameter group $G_6(A)$. Each of the ∞^4 triangles having A for one vertex is the invariant triangle of a two-parameter subgroup of $G_6(A)$. The groups $G_6(l)$ and $G_6(A)$ are dualistic groups. G_8 contains ∞^2 subgroups $G_6(l)$ and also ∞^2 subgroups $G_6(A)$.

Each of these groups contains, besides collineations of type I, also collineations of other types; but the collineations composing a group $G_6(l)$ or $G(A)$ are chiefly of type I. After all the types of collineations have been investigated, the structure of all these groups enumerated under type I will be examined in detail. (See §5.)

30. *The Group $G_5(Al)$.* The general projective group G_8 contains ∞^3 subgroups $G_5(Al)$, one for each lineal element in the plane. Each group $G_5(Al)$ is made up of ∞^3 two-parameter groups $G_2(ABC)$ whose invariant triangles have one vertex and one side in common. Every group $G_6(l)$ contains ∞^1 subgroups

$G_5(Al)$, one for each point on l; likewise each group $G_6(A)$ contains ∞^1 subgroups $G_5(Al)$, one for each line through A. The group $G_5(Al)$ is self-dualistic.

31. *The Groups $G_4(AB)$ and $G_4(ll')$.* The points AB and their join l form the invariant figure of a four-parameter group $G_4(AB)$. The group G_8 contains ∞^4 equivalent subgroups $G_4(AB)$, one for each pair of points in the plane. A group $G_4(AB)$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have the vertices A and B in common.

In like manner, G_8 contains ∞^4 equivalent subgroups $G_4(ll')$, one for each pair of lines in the plane. A group $G_4(ll')$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have one vertex and two sides in common. The groups $G_4(AB)$ and $G_4(ll')$ are dualistic. $G_6(l)$ contains ∞^2 subgroups $G_4(AB)$, also ∞^2 subgroups $G_4(ll')$; $G_6(A)$ breaks up in a similar manner.

32. *The Group $G_4(A, l')$.* There are ∞^4 combinations of point and line in the plane where the point is not on the line. Each combination of point and line is the invariant figure of a four-parameter group $G_4(A, l')$. The group $G_4(A, l')$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have a vertex and the opposite side in common. Thus we see that G_8 contains ∞^4 equivalent four-parameter subgroups $G_4(A, l')$; these groups are self-dualistic.

33. *The Group $G_3(ABl')$.* Triangles ∞^1 in number can be arranged with two of their vertices at A and B and their third vertex on a line Al'. Each of these triangles is the invariant triangle of a two-parameter group $G_2(ABC)$; these ∞^1 two-parameter groups unite to form a three-parameter group $G_3(ABl')$.

The group of all collineations in the plane G_8 contains ∞^5 equivalent subgroups $G_3(ABl')$; a group of this kind is self-dualistic. The group $G_4(AB)$ contains ∞^1 subgroups of the kind $G_3(ABl')$, one for each line through A; also one such group for each line through B. The group $G_4(ll')$ contains ∞^1 subgroups of the kind $G_3(ABl')$, one for each point on l', and also one for each point on l.

THEOREM 20.—*The group of all collineations in the plane G_8 contains eight subgroups which are made up of two-parameter groups of the kind $G_2(ABC)$; these are $G_6(A)$, $G_6(l)$; $G_5(Al)$; $G_4(A, l')$, $G_4(AB)$, $G_4(ll')$; $G_3(ABl')$; $G_2(ABC)$.*

C.—Groups of the Second Class.

34. *Groups in which r is Constant.* We now proceed to the determination of groups of another variety, viz. those in which every collineation is characterized by the same value of r. It has already been shown that the two-parameter group $G_2(ABC)$ breaks up into ∞^1 one-parameter subgroups, each subgroup being characterized by a constant value of r. If we take all types of groups of the first class and in these keep r constant and let the other parameters vary, we will sometimes find subgroups by this process and sometimes not. The groups G_8; $G_6(l)$, $G_6(A)$; $G_5(Al)$; $G_4(A, l'')$, $G_4(AB)$, $G_4(ll')$; $G_3(ABl')$ must each be examined separately.

35. *The Group $G_2(ABl')$ r.* The three-parameter group $G_3(ABl')$ is composed of ∞^1 two-parameter groups $G_2(ABC)$; for we can construct an infinity number of triangles having two of their vertices at A and B and the third vertex C on the line l'. Choose from each of these two-parameter groups the one parameter subgroup characterized by a certain value of r. It can be shown that the aggregate of all collineations belonging to these one-parameter groups characterized by the same constant r forms a two-parameter group. Let us take two collineations T and T_1, one from each of the groups $G_2(ABC)$ and $G_2(ABC_1)$, and find their resultant. The characteristic cross-ratios of T are k, k^{-r} and k^{r-1} along the sides AB, BC, and CA respectively; those of T_1 are k_1, k_1^{-r} and $k_1^{(r-1)}$ along the sides AB, BC_1, and C_1A respectively. The resultant of $T(ABC)$ and $T_1(ABC_1)$ is $T_2(ABC_2)$, where C_2 is some point on the line ACC_1 or l'. Along AB we have $k_2 = kk_1$ for the two component one-dimensional transformations have both invariant points in common. Along ACC_1 we have $k^{r-1}\, k_1^{r-1} = (kk_1)^{r-1} = k_2^{r-1}$; for the two one-dimensional transformations have a common invariant point A. The two one-dimensional transformations of the pencil through the point A have both invariant rays in common, and are characterized by k^{-r} and k_1^{-r} respectively, hence their resultant is given by $k_2^{-r} = (kk_1)^{-r}$ and hence the transformation along BC_2 is also given by k_2^{-r}. Therefore, the three characteristic cross-ratios of T_2 are k_2, k_2^{-r}, and k_2^{r-1}.

Since these two collineations T and T_1, from different one-parameter subgroups but characterized by the same value of r, compound into a collineation belonging to another one-parameter group but with the same value of r, it follows that all the collineations belonging to these one-parameter groups with the same r form

a two-parameter group $G_2(ABl')r$. There is one such two-parameter group within $G_3(ABl')$ for every value of r.

THEOREM 21.—*The group $G_3(ABl')$ contains ∞^1 two-parameter subgroups $G_2(ABl')r$, one for each value of r. The groups $G_3(ABl')$ and $G_2(ABl')r$ have the same invariant figure.*

36. *The Groups $G_3(ll')r$ and $G_3(AB)r$.* In like manner it can be shown that the four-parameter group $G_4(ll')$ can be split up into ∞^1 three-parameter groups such that every collineation belonging to one of these three-parameter subgroups is characterized by the same value of r. Consider ∞^2 triangles having common the sides l and l' and the vertex A. Each of these triangles determines a two-parameter group $G_2(ABC)$; take from each of these two-parameter groups the one-parameter group characterized by a certain value of r. These ∞^2 one-parameter groups form a three-parameter group every collineation in which is characterized by the same value of r. To prove this, take two collineations T from $G_2(ABC)$ and T_1 from $G_2(AB_1C_1)$. Let the three cross-ratios of T be k, k^{-r}, k^{r-1} and those of T_1 be k_1, k_1^{-r}, k^{r-1}. The resultant of T and T_1 is T_2. Along ABB_1 we have $k_2 = kk_1$, for the two one-dimensional transformations have the invariant point A in common; along ACC_1 we have $k_2^{r-1} = k^{r-1}k_1^{r-1}$ for the same reason. In the pencil through A we have $k_2^{-r} = k^{-r}k_1^{-r}$, for the two one-dimensional transformations have both invariant rays in common. Hence, the resultant T_2 has for its three cross-ratios k_2, k_2^{-r}, k_2^{r-1}.

Since the resultant of any two collineations of $G_4(ll')$ which are characterized by the same r is also characterized by the same r, it follows that all collineations in $G_4(ll')$ which have the same value of r form a subgroup $G_3(ll')r$. There is one such subgroup in $G_4(ll')$ for every value of r. The four variable parameters of $G_4(ll')$ are the segments AB and AC, k and r; if r be kept constant and the other three parameters be made to vary, we obtain the subgroup $G_3(ll')r$.

In a similar manner it may be shown that the group $G_4(AB)$ contains ∞_1 three-parameter subgroups each characterized by a constant value of r. To prove this, consider ∞^2 triangles having the two vertices in common and take the collineations $T(ABC)r$ and $T_1(ABC_1)r$; their resultant is $T_2(ABC_2)r$. The details of the proof are left for the reader to supply. The two groups $G_3(ll')r$ and $G_3(AB)r$ are dualistic to each other.

THEOREM 22.—*The group* $\left\{ \begin{matrix} G_4(ll') \\ G_4(AB) \end{matrix} \right\}$ *contains* ∞^1 *subgroups* $\left\{ \begin{matrix} G_3(ll')\,r \\ G_3(AB)\,r \end{matrix} \right\}$, *one for each value of* r.

37. *The Group* $G_4(Al)\,r$. The five parameter group $G_5(Al)$, whose invariant figure is the lineal element Al contains ∞^1 four-parameter subgroups each characterized by a constant value of r. To show this, take two collineations $T(ABC)\,r$ and $T_1(AB_1C_1)\,r$, having the same value of r, from the two-parameter groups $G_2(ABC)$ and $G_2(AB_1C_1)$ whose invariant triangles have the vertex A and the side l in common. The cross-ratios of T are k, k^{-r}, k^{r-1}; those of T_1 are k_1, k_1^{-r}, k_1^{r-1}. Along ABB_1 we have $k_2 = kk_1$, for these two one-dimensional transformations have the invariant point A in common. In the pencil through A we have $k_2^{-r} = k^{-r}k_1^{-r}$, for these two one-dimensional transformations have the ray ABB_1 in common. The third cross-ratio of the resultant T_2 must be k_2^{r-1}, for the product of the three cross-ratios taken in order around the triangle must be unity. The three cross-ratios of T_2 are, therefore, k_2, k_2^{-r}, k_2^{r-1} and its invariant triangle is AB_2C_2 where B_2 is on the line ABB_1. Thus the resultant of any two collineations in $G_5(Al)$, both characterized by the same value of r, is also characterized by the same r. Thus we see that the group $G_5(Al)$ contains ∞^1 subgroups $G_4(Al)\,r$, one for each value of r.

THEOREM 23.—*The group* $G_5(Al)$ *contains* ∞^1 *four-parameter subgroups* $G_4(Al)\,r$, *one for each value of* r. *These subgroups all have in common the same invariant lineal element.*

38. *Subgroups of* $G_4(A, l')$, $G_6(l)$, $G_6(A)$, G_8. It is easy to show that the remaining groups of the first class, viz. $G_4(A, l')$, $G_6(l)$, $G_6(A)$, and G_8 do not break up into subgroups in the above manner. For example, combine two collineations T and T_1, characterized by the same value of r, from $G_6(l)$; the invariant triangles of T and T_1 have the line l for a common side, but no common vertex. Along the invariant side l the law of combination $k_2 = kk_1$ does not hold, because the two one-dimensional transformations have no invariant point in common; hence, there is no reason why the resultant T_2 should be characterized by the same r as T and T_1. Like reasons hold in the case of the other three groups $G_4(A, l')$, $G_6(A)$ and G_8; thus, in general, the three groups $G_4(A, l')$, $G_6(A)$, and $G_6(l)$ do not break up into subgroups characterized by a constant r. There is, however,

one exceptional value of r for which these three groups each yield one subgroup. This exceptional case will be discussed later. See articles 44 and 45.

39. *Critical Values of r.* The six critical values of r, viz. $r = 1, 0, \infty, -1, 2, 1/2$ are to be excepted for each of these groups of the second class. For $r = 1, 0, \infty$ the collineations are all perspective collineations and the groups corresponding to these values of r are groups of perspective collineations. For $r = -1, 2, 1/2$ the path-curves are conics and the resulting groups belong to the third class to be considered below.

THEOREM 24.—*There are five, and only five, varieties of groups of the second class, viz.* $G_1(ABC)r$, $G_2(ABl')r$, $G_3(AB)r$, $G_3(ll')r$, $G_4(Al)r$; *of these the two groups* $G_3(AB)r$ *and* $G_3(ll')r$ *are dualistic and the other three are self-dualistic.*

D.—Groups of the Third Class.

40. *Groups whose Path-Curves are Conics.* We come now to the consideration of the groups of collineations of type I which are made up of one-parameter groups whose path-curves are conics. The groups which we shall find are, in many instances, only special cases of the groups $G_2(ABl')r$, $G_3(AB)r$, $G_3(ll')r$ and $G_4(Al)r$ when $r = -1, 2, 1/2$; but we shall find many groups which are not special cases of the above, and hence it is best, even at the risk of being tedious, to enumerate all varieties of groups of this class.

There is only one variety of one-parameter group of this kind; in this case the two common tangents to the conics and the common chord of contact form the invariant triangle. This group will be symbolized by $G_1(K_n)$.

41. *The Groups $G_3(K)$ and $G_2(AKl)$.* A very important group $G_3(K)$, with invariant conic K, can be built up as follows: ∞^2 triangles can be constructed consisting of two tangents to K and their chord of contact. Belonging to each such triangle is a one-parameter group $G_1(K_n)$ having K among its system of invariant conics. These ∞^3 collineations, leaving K invariant, form a three-parameter group $G_3(K)$ by the principle that the aggregate of all collineations, leaving a given figure invariant, forms a group.

The group $G_3(K)$ contains ∞^1 two-parameter subgroups which may be constructed as follows: take a conic K and a tangent l to it at a point A of the

conic. We can construct ∞^1 triangles having one vertex at A, another vertex at B on l, and a third vertex C on the conic such that AB and BC are tangents to K, and AC is the chord of contact. All one-parameter groups belonging to these triangles, whose path-curves are conics touching AB at A, have K among their invariant path-curves. Hence, these collineations, leaving invariant K, a point on K, and the tangent to K at the point, form a two-parameter group $G_2(AKl)$. The two variable parameters of this group are the cross-ratio k and the position of the point C on the conic K.

THEOREM 25.—*All collineations of the plane which leave a conic K invariant, form a three-parameter group $G_3(K)$; this group contains ∞^1 two-parameter subgroups $G_2(AKl)$, one for each point on K.*

The groups $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$ and $G_4(Al)\,r$, where $r = -1$, 2, 1/2 are all special cases of those of the second class. Three of these groups of the same order differ only in the arrangement of the invariant triangles of the one-parameter subgroups $G_1(K_n)$.

42. *The Group $G_3(S)$.* There is still another three-parameter group of collineations composed of one-parameter groups $G_1(K_n)$. Take a lineal element Al and a pencil of conics S all touching l at A and having at A contact of the third order with one another. One property of this pencil of conics is that the polars of a point on l with respect to the pencil S coincide, and this common polar passes through A. A pencil of conics having contact of the third order is projectively transformed into a pencil of the same kind. We can construct ∞^2 triangles having one line AB along l, one vertex at A, and one vertex at C on one of these conics, so that BC is a tangent to the curve at C, and AC is the chord of contact. Belonging to each of these triangles is a one-parameter group $G_1(ABC)\,r = -1$ such that one conic of the pencil S is included in its family of path-curves. Every collineation in such a one-parameter group leaves invariant one of the conics of S and interchanges the other conics of S; hence, it leaves invariant the pencil S as a whole and the lineal element Al. The aggregate of all collineations, leaving S invariant, forms a three-parameter group $G_3(S)$.

Since the group $G_3(S)$ leaves invariant the lineal element Al, and the pencil of conics S having contact of the third order with Al, we see that $G_3(S)$ can be built up out of ∞^1 two-parameter groups $G_2(AKl)$, one for each conic in S.

Again, since the polars of any point B on l with respect to each conic of S coincide in l' a line through A, we see that $G_3(S)$ is composed of ∞^1 subgroups $G_2(ABl')\, r = 1$.

43. *The Structure of* $G_4(Al)\, r = -1$. The group $G_4(Al)\, r = -1$ breaks up into subgroups in several different ways, and accordingly its structure is peculiar and worthy of special attention.

There are ∞^3 distinct conics touching l at A. This system of conics is composed of ∞^1 nets; each net contains ∞^2 conics having contact of the second order at A, and hence all conics of the net have a common circle of curvature at A. Each of the circles touching l at A is the circle of curvature of such a net.

Each net N is composed of ∞^1 pencils S such that the conics of each pencil have contact of the third order at A. The polar of the point at infinity on l with respect to the conics of the pencil S is a line l' through A, the line of centres of S. Each line through A is the line of centres of one of the pencils of the net N. Since there are ∞^1 nets N, each line through A is the line of centres of ∞^1 pencils S. The system of conics touching l at A is, therefore, composed of ∞^2 pencils S.

Each pencil S is the invariant figure of a group $G_3(S)$, hence, there are ∞^2 such groups all contained within the group $G_4(Al)\, r = -1$. But since $G_4(Al)\, r = -1$ contains ∞^4 collineations and ∞^2 subgroups $G_3(S)$, it seems that two such subgroups as $G_3(S)$ and $G_3(S_1)$ must contain certain collineations in common. Let us consider two pencils of conics S and S_1 which do not belong to the same net N. A given conic K of the pencil S is cut by each conic of S_1 in two points other than A. When the two points of intersection of K and K_1 coincide, the conics K and K_1 have double contact. Each conic of S has double contact with one, and only one, conic of S_1. We wish to find the locus of the points of contact of corresponding conics of S and S_1.

The polars of a point B on l with respect to all conics of S is the same line l' through A. Thus the range of points on l and the pencil of lines through A are projectively related by means of S. In the same manner, by means of S_1, the range of points on l is projectively related to the pencil of lines through A. These two pencils of lines being projective to one another, have two self-corresponding rays. One of these is l, since l is the polar of A with respect to both S and S_1. The other self-corresponding ray is a line l' through A. Hence, the

polars of a certain point B on l, with respect to both S and S_1, coincide in l', and the locus of the points of contact of conics of S and S_1 is l', a straight line through A. Therefore, we see that the two three-parameter groups $G_3(S)$ and $G_3(S_1)$ contain the same two-parameter group $G_2(ABl')\,r = -1$.

On the other hand, if the two pencils S and S_1 belong to the same net N of conics having contact of the second order at A, the conics from S and S_1 cannot have double contact, and hence two such groups $G_3(S)$ and $G_3(S_1)$ have no collineations in common.

In a certain net N there are ∞^1 pencils S, S_1, S_2, etc.; each of these pencils is the invariant pencil of a group $G_3(S_n)$. ∞^1 such groups $G_3(S_n)$, no two of which contain a collineation of type I in common, include all collineations of this type in $G_4(Al)\,r = -1$. It is easy to see that the ∞^3 subgroups $G_1(ABC)\,r = -1$ in $G_4(Al)\,r = -1$ are all included in the ∞^1 groups $G_3(S_n)$ of the net N. If it be true that all collineations of type I in $G_3(Al)\,r = -1$ are included in the ∞^1 groups $G_3(S_n)$ of the net N, then all collineations of type I in a group $G_3(S')$, where S' is a pencil of conics not included in N, are to be found in the ∞^1 groups $G_3(S_n)$ of the net N. In fact, if we take $G_3(S')$ in turn with each of the groups $G_3(S_n)$ of the net, we see that $G_3(S')$ has a two-parameter subgroup $G_2(ABl')\,r = -1$ in common with each group $G_3(S_n)$ of the net; the common subgroup $G_2(ABl')\,r = -1$ is different for each of the groups $G_3(S_n)$ of the net. In this way it can be shown without difficulty that every collineation of type I in $G_3(S')$ is also to be found in the net of groups $G_3(S_n)$.

THEOREM 26.—*The group $G_4(Al)\,r = -1$ contains ∞^2 subgroups $G_3(S)$; two subgroups $G_3(S)$ and $G_3(S_1)$ contain no collineation of type I in common when the two pencils are S and S_1 belong to the same net N; they have a subgroup $G_2(ABl)\,r = -1$ in common when S and S_1 belong to different nets. Each group $G_3(S)$ contains ∞^1 subgroups $G_2(AKl)$, one for each conic in S.*

44. *The Group $G_3(A, l')\,r = 2$.* It was remarked in article 38 that the group $G_4(A, l')$ does not break up into subgroups $G_3(A, l')\,r$ for all values of r; but that one such subgroup exists, viz. $G_3(A, l')\,r = 2$. In order to prove this, we take two collineations $T(ABC)\,r$ and $T_1(AB_1C_1)\,r$ characterized by the same value of r. Their invariant triangles (ABC) and (AB_1C_1) have the vertex A and the side l'' opposite A in common. The cross-ratios of T are k, k^{-r}, k^{r-1} along AB, BC and CA respectively; the cross-ratios of T_1 are k_1, k_1^{-r} and k_1^{r-1}

along AB_1, B_1C_1, C_1A respectively. The resultant of T and T_1 is T_2 and leaves invariant a triangle AB_2C_2, where B_2 and C_2 are some pair of points on the line BC or l''.

Introduce now an auxiliary collineation U, leaving invariant the triangle ABB_1 and characterized by λ and r. We can show that the value of λ may be so chosen that the resultant of T and U leaves invariant the triangle ABB_2, while the resultant of U^{-1} and T_1 leaves invariant the triangle AB_1C_2.

Since the invariant triangles of T and U are respectively ABC and ABB_1, the resultant TU of T and U will leave invariant a triangle ABX, where X is some point on l''. Since the invariant triangles of U^{-1} and T_1 are respectively AB_1B and AB_1C_1, the resultant $U^{-1}T$ of U^{-1} and T will leave invariant a triangle AB_1X_1, where X_1 is some point on l''. If λ be made to vary through all values, the points X and X_1 will each assume in turn all positions on the line l''. The two projective ranges, thus generated by X and X_1, will have two self-corresponding or double points, D and D_1, in common. Let λ_1 be that value of λ for which X and X_1 coincide in D, and let U be so chosen that its characteristic constants are λ_1 and r. Since the invariant triangles of TU and $U^{-1}T_1$ are now AD_1B and AD_1B_1, the invariant triangle of their resultant will be AD_1D'. But this resultant is $TUU^{-1}T = TT_1$ whose invariant triangle is AB_2C_2. Hence, the point D' is either B_2 or C_2. In like manner, by choosing for λ a value λ_2 for which X and X_1 coincide at D_1, it may be shown that D_1 is either B_2 or C_2. Hence, λ may always be so chosen, and in two distinct ways, so that the resultant of T and U leaves invariant the triangle ABB_2, while the resultant of U^{-1} and T_1 leaves invariant the triangle AB_1C_2.

The collineations T and T_1, U and U^{-1}, with their cross-ratios, may be written symbolically as follows:

$$T(ABC) = \begin{Bmatrix} k & k^{-r} & k^{r-1} \\ AB, & BC, & CA \end{Bmatrix}; \quad T_1(AB_1C_1) = \begin{Bmatrix} k_1 & k_1^{-r} & k_1^{r-1} \\ AB_1, & B_1C_1, & C_1A \end{Bmatrix},$$

$$U(ABB_1) = \begin{Bmatrix} \lambda & \lambda^{-r} & \lambda^{r-1} \\ AB, & BB_1, & B_1A \end{Bmatrix}; \quad U^{-1}(ABB_1) = \begin{Bmatrix} \lambda^{-1} & \lambda^{r} & \lambda^{1-r} \\ AB, & BB_1, & B_1A \end{Bmatrix}$$

or

$$U^{-1}(AB_1B) = \begin{Bmatrix} \lambda^{r-1} & \lambda^{-r} & \lambda \\ AB_1, & B_1B, & BA \end{Bmatrix}.$$

Since the invariant triangles of T and U have two vertices in common, the parameters k and k_1 combine according to the law $kk_1 = k_2$; the same is true for

U^{-1} and T_1, and also for TU and $U^{-1}T_1$. Hence, we have

$$TU(ABB_2) = \begin{Bmatrix} (k\lambda) & (k\lambda)^{-r} & (k\lambda)^{r-1} \\ AB & BB_2 & B_2A \end{Bmatrix}$$

or

$$TU(AB_2B) = \begin{Bmatrix} (k\lambda)^{1-r} & (k\lambda)^{r} & (k\lambda)^{-1} \\ AB_2 & B_2B & BA \end{Bmatrix}$$

and

$$U^{-1}T_1(AB_1B_2) = \begin{Bmatrix} (k_1\lambda^{r-1}) & (k_1\lambda)^{-r} & (k_1^{r-1}\lambda) \\ AB_1 & B_1B_2 & B_2A \end{Bmatrix}$$

or

$$U^{-1}T_1(AB_2B_1) = \begin{Bmatrix} (k_1^{1-r}\lambda^{-1}) & (k_1\lambda) & (k_1^{-1}\lambda^{1-r}) \\ AB_2 & B_2B_1 & B_1A \end{Bmatrix}.$$

The resultant of TU and $U^{-1}T_1$ is $TUU^{-1}T_1 = TT_1$, and may be written

$$TT_1(AB_2C_2) = \begin{Bmatrix} (kk_1)^{1-r}\lambda^{-r} & (kk_1\lambda^2)^{r} & (kk_1)^{-1}\lambda^{-r} \\ AB_2 & B_2C_2 & C_2A \end{Bmatrix}.$$

From this we see that the collineation TT_1 is not, in general, characterized by the same r which characterizes T and T_1. For this to be true, the cross-ratios along B_2C_2 and C_2A must be equal to that along AB_2 raised to the power $-r$ and $r-1$ respectively; i. e. we must have the two equations

$$(kk_1\lambda^2)^r = [(kk_1)^{1-r}\lambda^{-r}]^{-r} \text{ and } (kk_1)^{-1}\lambda^{-r} = [(kk_1)^{1-r}\lambda^{-r}]^{r-1}.$$

All these conditions are fully satisfied when, and only when, $r^2 = 2r$; i. e. when $r = 0$ or $r = 2$. But if $r = 0$, T and T_1 are both perspective collineations, and their resultant is obviously also a perspective collineation. Hence, for collineations of type I we have only the result $r = 2$.

Therefore, the resultant of two collineations T and T_1, both taken from the group $G_4(A, l'')$ and characterized by the same value of r, is not a collineation characterized by the same r except when $r = 2$. Hence, the group $G_4(A, l'')$ does not break up into three-parameter subgroups, one for each value of r. But all collineations in $G_4(A, l'')$, for which $r = 2$, do form a three-parameter group $G_3(A, l'')\,r = 2$. In this group k_2 is not equal to the product of k and k_1.

THEOREM 27.—*The group $G_4(A, l'')$ contains one, and only one, three-parameter subgroup of the form $G_3(A, l'')\,r$, viz. $G_3(A, l'')\,r = 2$.*

45. *The Groups $G_5(l)\,r = 2$ and $G_5(A)\,r = 2$.* We shall now show in a manner similar to the above that the group $G_6(l)$ does not break up into sub-

groups $G_5(l)r$ for all values of r, but that one such subgroup exists for $r=2$. Take two collineations T and T_1 thus:

$$T(ABC)\equiv k,\ k^{-r},\ k^{r-1} \text{ and } T_1(A_1B_1C_1)\equiv k_1,\ k_1^{-r},\ k_1^{r-1},$$

in which the triangles ABC and $A_1B_1C_1$ have one side l' but no vertex in common; the resultant of T and T_1 is $TT_1(A_2B_2C_2)$, where B_2 and C_2 are a pair of points on l'. Introduce a collineation U and its inverse U^{-1} as follows:

$$U(OB_1C)\equiv\lambda,\ \lambda^{-r},\ \lambda^{r-1},\quad U^{-1}(OB_1C)\equiv\lambda^{-1},\ \lambda^{r},\ \lambda^{1-r},$$

where O is the intersection of A_1B_1 and AC. The resultants of T and U, U^{-1} and T_1 are as follows:

$$TU(XB_1C)\equiv(k\lambda),\ (k\lambda)^{-r},\ (k\lambda)^{r-1};$$

$$U^{-1}T_1(X_1B_1B_2)\equiv(k_1\ \lambda^{-1}),\ (k_1\lambda^{-1})^{-r},\ (k_1\lambda^{-1})^{r-1}$$

or

$$U^{-1}T_1(X_1B_2B_1)\equiv(k_1\lambda^{-1})^{1-r},\ (k_1\lambda^{-1})^{r},\ (k_1\lambda^{-1})^{-1}.$$

Here λ is so chosen that the invariant triangles of TU and $U^{-1}T_1$ have one vertex B_2 in common. The possibility of this may be shown as in last article; the resultant of TU and $U^{-1}T_1$ is evidently TT_1.

In combining $TU(XB_2C)$ and $U^{-1}T_1(X_1B_2B_1)$, we see that along the side BC or l' the two one-dimensional transformations have the invariant point B_2 in common; through the point B_2 the two one-dimensional transformations of the pencil have the invariant ray l' in common. Thus, in both these cases, the law $kk_1=k_2$ holds good. Hence, we have

$$TT_1(A_2B_2C_2)\equiv\left\{\begin{matrix}kk_1^{1-r}\lambda^r, & k^{-r}k_1^r\lambda^{-2r}, & k^{r-1}k_1^{-1}\lambda_r\\ A_2B_2, & B_2C_2, & C_2A_2\end{matrix}\right\}.$$

In this resultant collineation the cross-ratio along B_2C_2 is not that along A_2B_2 raised to the power $-r$ except for $r=2$ or 0. Thus, if $(kk_1^{1-r}\lambda^r)^{-r}=k^{-r}k_1^r\lambda^{-2r}$, we must have

$$k_1^{r^2-r}\lambda^{-r^2}=k_1^r\lambda^{-2r}.$$

This is satisfied when $r^3=2r$, i. e. when $r=2$ or 0. But if $r=0$, T and T_1 are both perspective collineations. Hence, for collineations of type I, we have only the result $r=2$.

It may be shown in a similar manner that the resultant of two collineations T and T_1, taken from $G_6(A)$ and characterized by the same value of r, is also

characterized by this value of r only when $r = 2$. The proof is left for the reader.

THEOREM 28.—*The group* $\left\{\begin{matrix} G_6(l) \\ G_6(A) \end{matrix}\right\}$ *contains one, and only one, five-parameter subgroup of the kind* $\left\{\begin{matrix} G_5(l)\, r \\ G_5(A)\, r \end{matrix}\right\}$, *viz. the group* $\left\{\begin{matrix} G_5(l)\, r = 2 \\ G_5(A)\, r = 2 \end{matrix}\right\}$.

46. *Summary of Groups of Third Class.* We have thus found eleven varieties of groups of the third class. Five of these eleven groups, viz. $G_1(ABC)\, r$, $G_2(AB l')\, r$, $G_3(AB)\, r$, $G_3(ll')\, r$, and $G_4(Al)\, r$, where $r = -1,\ 2,\ 1/2$, are only special cases of groups of the second class. On the other hand, the six varieties of groups, viz. $G_2(AKl)$, $G_3(K)$, $G_3(S)$, $G_3(A, l'')\, r = 2$, $G_5(l)\, r = 2$ and $G_5(A)\, r = 2$ are essentially of the third class.

THEOREM 29.—*There are six distinct varieties of groups of the third class, viz.* $G_2(AKl)$, $G_3(K)$, $G_3(S)$, $G_3(A, l'')\, r = 2$, $G_5(l)\, r = 2$ *and* $G_5(A)\, r = 2$.

§3.—COLLINEATIONS OF TYPE II.

A. The Group $G_2'(AB l')$ *and its Subgroups.*

47. *The Properties of a Collineation* T'. The kind of collineation which we shall call type II is most easily obtained as the resultant of a perspective collineation $S(B, l')$ and an elation $S'(Al)$. Let the invariant figures of S and S' be so chosen that B, the vertex of S, lies on l the axis of S', but not at A; also let l', the axis of S, pass through A the vertex of S', and let ϕ be the angle between l and l'. Let the cross-ratio of S be k and the constant of S' be α. The resultant of S and S' leaves invariant the points A and B and the lines l and l'. This resultant will be designated by T' and its invariant figure by $(AB l')$.

Along the line l, S' produces an identical transformation i and S produces a one-dimensional transformation t, whose invariant points are A and B, and whose cross-ratio is k; their resultant is t. Along l', S produces an identical transformation and S' a one-dimensional transformation t', whose single invariant point is A, and whose constant is $\alpha \sin \phi$; their resultant is t'. Through the point A S' produces an identical transformation i, and S a one-dimensional transformation t_1, with two invariant rays l and l', and whose cross-ratio is k; their

resultant is t_1. Through B, S produces an identical transformation i and S' a one-dimensional transformation t_1', with a single invariant ray l and a constant $a\alpha$, where a is the distance AB; their resultant is t_1'. Hence T' produces along AB and through A, one-dimensional transformations with two invariant elements and the same cross-ratio k, and along l' and through B, one-dimensional transformations with a single invariant element.

If the cross-ratio along AB be k, in the direction from A to B, then that through A is also k in the direction from l' to l. The relation of the constants of the transformations along l' and through B is found as follows: Let the constant along l' be α' and that through B be β; then $\alpha' = \alpha \sin\phi$ and $\beta = a\alpha$. Hence, $\alpha' = \frac{\sin\phi}{a}\beta$. Hereafter we shall designate the constant along l' by α and that through B by β, implying the relation

$$\alpha = \frac{\sin\phi}{a}\beta.$$

48. *The Group $G_2'(ABl')$.* A collineation T' is completely characterized by its invariant figure (ABl') and two independent constants k and α. It is evident that if k and α be made to vary continuously, we have ∞^2 collineations T' with the same invariant figure (ABl'). Two of these collineations, $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, combine to form $T_2'(k_2, \alpha_2)$, where $k_2 = kk_1$ and $\alpha_2 = \alpha + \alpha_1$. T_2' has the same invariant figure as T' and T_1'. These ∞^2 collineations, leaving (ABl') invariant, form a two parameter group $G_2'(ABl')$, the parameters being k and α.

THEOREM 30.—*All collineations of type II, which have the same invariant figure (ABl'), form a two-parameter group $G_2'(ABl')$.*

49. *One-Parameter Groups $G_1'(ABl')\,a$.* We shall now go on to show that the group $G_2'(ABl')$ contains ∞^1 one-parameter subgroups. The group $G_2'(ABl')$ depends on two independent parameters, k and α. Let us assume a relation between these quantities and consider only those collineations of the group $G_2'(ABl')$ for which k and α satisfy the assumed relation. Let $k = a^\alpha$, where a is some constant.

Take two collineations, $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, belonging to $G_2'(ABl')$ and combine them to form $T_2'(k_2, \alpha_2)$; let $k = a^\alpha$ and $k_1 = a_1^{\alpha_1}$. We have $\alpha_2 = \alpha + \alpha_1$ and $k_2 = kk_1$, i. e. $a_2^{\alpha_2} = a^\alpha a_1^{\alpha_1} = a^{\alpha+\alpha_1}\ (a_1/a)^{\alpha_1}$. If we put $a_1 = a$, we get

$k_2 = a_2^{a_2} = a^{a_2}$; whence we see that $a_2 = a_1 = a$. Therefore two collineations T'' and T_1', both characterized by the same value of a, result is a collineation also characterized by the same a. From this we see that all collineations of the group $G_2'(ABl')$, for which the parameters k and α satisfy the relation $k = a^\alpha$, form a subgroup of $G_2'(ABl')$. This subgroup $G_1'(ABl')a$ is characterized by a constant a, and has only one parameter α; it is therefore a one-parameter subgroup of $G_2'(ABl')$. Within $G_2'(ABl')$ there are ∞^1 such subgroups, one for each value of a (except $a = 0$).

50. *The Path-Curves of* $G_1'(ABl')a$. The effect upon a point of the plane of all the collineations of the group $G_1'(ABl')a$, is to move it along its path-curve. The equation of the family of path-curves of the group $G_1'(ABl')a$ is readily obtained as follows: In the fundamental figure ABl', let the line AB be the axis of x and l' the axis of y, so that A is the origin of a system of cartesian coordinates, generally oblique. The collineation $T'(a^\alpha, a)$ transforms the point (x, y) to (x_1, y_1). The cross-ratio a^α of the pencil through A is given by $y_1/x_1 = a^\alpha y/x$; the parameter of the pencil through B is given by $(x_1 - c)/y_1 = (x - c)/y + \beta$, $(\cot\theta_1 = \cot\theta + \beta)$, c being the distance AB. Eliminating α from these equations, we have, putting $b = a^{\frac{\sin\phi}{c}}$,

$$\frac{y_1/x_1}{b^{\frac{x_1-c}{y_1}}} = \frac{y/x}{b^{\frac{x-c}{y}}} = C, \text{ whence } y = Cxb^{\frac{x-c}{y}}.$$

This represents the path-curves of the group $G_1'(ABl')a$, C being the parameter of the family of curves. These curves all pass through A and B and have these points as singular points.

THEOREM 31.—*The two-parameter group* $G_2'(ABl')$ *contains* ∞^1 *one-parameter subgroups* $G_1'(ABl')a$; *each subgroup is characterized by a constant* a *and a parameter* α. *The path-curves of* $G_1'(ABl')a$ *are given by* $y = Cxa^{\frac{x-c}{y}}$

51. *Two Special Subgroups of* $G_2'(ABl')$. Among the ∞^1 subgroups of $G_2'(ABl')$, two are of particular importance, and require special attention. When $a = 1$, we have $k = 1^\alpha = 1$ for all values of α. This signifies that the one-dimensional transformations along AB and through A are both identical transfor-

mations, and hence all points on l and all lines through A are invariant under all collineations of the group. This particular subgroup of $G_2'(ABl')$ is, therefore, the group of elations $H_1'(Al)$.

When $a = 0$ and k varies, this is equivalent to $a = \infty$; the one-dimensional transformations along l' and through B are both identical. In this case, all points on l' and all lines through B are invariant under all collineation of the group. This subgroup of $G_1'(ABl')$ is, therefore, the group of perspective collineations $H_1(B, l')$.

Every collineation T' in the group $G_2'(ABl')$ is the resultant of an elation from the group $H_1'(Al)$ and a perspective collineation from the group $H_1(B, l')$. In fact, if we combine every elation in $H_1'(Al)$ with every perspective collineation in $H_1(B, l')$, we obtain ∞^2 collineations T' which constitute the group $G_2'(ABl')$.

THEOREM 32.—*The two-parameter group $G_2'(ABl)$ contains one subgroup of elations $H_1'(Al)$ and one subgroup of perspective collineations $H_1(B, l')$.*

52. *Properties of the Group $G_1'(ABl')a$.* The parameter of the group is a, and the law of combination of parameters is $a_2 = a + a_1$. Consequently, the properties of the group $G_1'(ABl')a$ are quite similar to those of the one-parameter group of elations. The identical collineation of the group is given by $a = 0$, pseudo-collineations are given by $a = \infty$ and $k = 0$. The group contains ∞^1 infinitesimal collineations corresponding to the infinitesimal values of a; hence the group is continuous everywhere except at $a = \infty$. The groups $G_1'(ABl')a$ and $G_1'(ABl')a_1$ contain no collineations in common save the identical and the pseudo-collineations.

53. *Classification of the Groups of Type II.* We are now in position to determine the number of collineations of type T' in the plane. Since there are ∞^5 different figures of the kind (ABl') in the plane, it follows that there are ∞^7 collineations of the kind T'; but it does not follow that these form a seven-parameter group. In general, two collineations T and T_1' having no parts of their invariant figures in common, do not result in a collineation of type II but of type I.

Our next problem is to determine all types and varieties of groups composed chiefly of collineations of type II. We shall find two distinct classes of such

groups, viz. (1) groups made up of two-parameter groups $G_2'(ABl')$, and (2) groups made up of collineations, all of which are characterized by the same value of a. In groups of the first class a and α both vary, and in groups of the second class a is a constant and α a variable.

B.—Groups of Type II of the First Class.

54. *The Groups $G_3'(AB)$ and $G_3'(ll')$.* Take two groups G_2' whose invariant figures are (ABl') and (AB_1l'), the points A, B, and B_1 being collinear. Take two collineations $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, one from each group, and find their resultant $T_2'(k_2, \alpha_2)$. From the position of the invariant figures, we see that $k_2 = kk_1$ and $\alpha_2 = \alpha + \alpha_1$. Hence, the invariant figure of T_2' is (AB_2l'), B_2 being some point on AB, and the resultant collineation is of type II. Thus all the collineations in the two parameter groups G_2', whose invariant figures are the same, except that the point B is in different positions on the line AB, form a three-parameter group $G_3'(ll')$.

In a similar manner, it may be shown that all the collineations in the two-parameter groups G_2', whose invariant figures are the same, except that the line l' is in different positions through the point A, form a three-parameter group $G_3'(AB)$ dualistic to $G_3'(ll')$.

55. *The Group $G_4'(Al)$.* Again let us take two collineations $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, one from each of the groups $G_2'(ABl')$ and $G_2'(AB_1l_1')$ whose invariant figures have common the point A and the line l, but not the same line l' or the point B, and form their resultant $T_2'(k_2, \alpha_2)$. The one-dimensional transformations along AB, since they have the point A in common, combine according to the law $k_2 = kk_1$; also the one-dimensional transformations of the pencil through A, since they have the ray l in common, combine according to the law $k_2 = kk_1$. The resultant collineation, $T_2'(k_2, \alpha_2)$, is, therefore, of type II and its invariant figure is (AB_2l_2'). Hence, the collineations in the two parameter groups G_2', whose invariant figures are (AB_nl_n') (where B_n is in turn every point on l and l_n' is in turn every line through A), form a four-parameter group $G_4'(Al)$, whose invariant figure is the lineal element (Al). This group is self-dualistic for the reason that the figure (AB_nl_n) is a self-dualistic figure.

56. *The Group $G_4'(Al')$.* There are two other varieties of four-parameter groups, dualistic to each other, which can be formed from two-parameter groups

$G_2'(AB\overline{l}')$. Take two groups G_2' whose invariant figures are $(AB\overline{l}')$ and $(AB_1\overline{l}')$, having the line $\overline{l}'$ and the point A in common; take a collineation from each group and combine them. The two one-dimensional transformations of the kind t' along $\overline{l}'$ result in a transformation t' of the same kind, leaving invariant A and $\overline{l}'$. The two transformations of the pencil through A, since they have the ray $\overline{l}'$ in common, combine according to the law $k_2 = kk_1$, and result in a transformation of the same kind, leaving invariant A and $\overline{l}'$ and some second line l_2 through A. The principle of duality requires that the transformation along l_2 should be of the kind t with two invariant points, hence, there is a second invariant point B_2 somewhere on l_2. The resultant collineation T_2' is, therefore, of type II and its invariant figure is $(AB_2\overline{l}')$. Hence, the collineations contained in the ∞^2 two-parameter groups G_2', whose invariant figures are (AB_nl), where B_n is in turn every point in the plane, form a four-parameter group $G_4'(Al')$. This group is evidently of a different variety from that last discussed.

57. *The Group of* $G_4'(Bl)$. The group dualistic to the above is composed of ∞^2 two-parameter groups G_2' whose invariant figures have in common the point B and the line l. Take two collineations T and T_1', one from each of the groups $G_2'(AB\overline{l}')$ and $G_2'(A_1Bl_1')$, and combine them. The two one-dimensional transformations of the kind t' of the pencil through B result in a transformation of the same kind; the two transformations of the kind t along l, since they have the point B in common, result in a transformation of the same kind according to the law $k_2 = kk_1$. The resultant of T' and T_1' is, therefore, of type II, and all the collineations belonging to the above-mentioned two-parameter groups unite to form a four-parameter group $G_4'(Bl)$.

58. *Summary of Groups of the First Class.* We have now enumerated all groups of this class that can be compounded out of the ∞^5 two-parameter groups $G_2'(ABl)$. For example, ∞^3 two-parameter groups G_2' can be so chosen that all collineations in them leave the line l invariant; but the resultant of any two collineations taken from different groups G_2', which have no invariant point in common, is no longer of type II but of type I. Hence, these ∞^3 groups G_2' do not form a five-parameter group.

THEOREM 33.—*There are six varieties of groups of collineations of type II in which the collineations of a group are not all characterized by a constant a, viz.* $G_2'(AB\overline{l}')$, $G_3'(AB)$, $G_3'(l\overline{l}')$, $G_4'(Al)$, $G_4'(Al')$, $G_4'(Bl)$.

C.—Groups of Type II of Second Class.

Returning now to the group $G_1'(ABl')a$ and using it as the fundamental group, it is possible to build up another series of groups of higher order; we shall find four varieties of such groups.

59. *The Groups $G_2'(AB)a$ and $G_2'(ll')a$.* The three-parameter group $G_3'(AB)$ is composed of ∞^1 two-parameter groups $G_2'(ABl)$, and hence contains ∞^2 one-parameter groups $G_1'(ABl')a$. Among these ∞^2 subgroups of $G_3'(AB)$, consider the ∞^1 groups G_1' characterized by the same value of a. Take two collineations $T'(a^\beta, \beta)$ and $T_1'(a^{\beta_1}, \beta_1)$ from different one-parameter groups and combine them. The two transformations of the pencil through B are both of the kind t' and have an invariant ray in common; hence, their resultant is of the same kind and $\beta_2 = \beta + \beta_1$. The two transformations along AB and through A are in both instances of the kind t, and combine according to the law $k_2 = kk_1 = a^\beta\, a^{\beta_1} = a^{\beta_2}$. Therefore, the resultant T_2' is of type II and is characterized by the same value of a as T' and T_1'. Hence, all collineations of type II, characterized by the same value of a and leaving invariant two points A and B and their joining line l, form a two-parameter group $G_2'(AB)a$.

In a similar manner, it may be shown that all collineations of type II, characterized by the same value of a and leaving invariant two lines and their point of intersection, form a two-parameter group $G_2'(ll')a$ dualistic to $G_2'(AB)a$.

60. *The Groups $G_3'(Al')a$ and $G_3'(Bl)a$.* The four-parameter group $G_4'(Al')$ or $G_4'(AB_nl')$, where B_n is in turn every point in the plane, is composed of ∞^3 one-parameter groups $G_1'(ABl')a$; ∞^2 of these one-parameter groups are characterized by the same value of a. Take from two of these groups two collineations $T'(a^\alpha, \alpha)$ and $T_1'(a^{\alpha_1}, \alpha_1)$, characterized by the same value of a, and combine them. The two transformations of the kind t' along l' combine to form one of the same kind according to the law $\alpha_2 = \alpha + \alpha_1$. The two transformations of the kind t through A, since they have the ray l' in common, combine to form one of the same kind according to the law $k_2 = kk_1 = a^\alpha\, a^{\alpha_1} = a^{\alpha_2}$. Hence their resultant T_2' is also of type II and is also characterized by the same value of a. Therefore, all the collineations in the ∞^2 one-parameter groups in $G_4'(AB_nl')$, which are characterized by the same value of a, form a three-parameter group $G_3'(AB_nl')a$ or $G_3'(Al')a$. There is one such subgroup in $G_4'(Al')$ for each value of a.

In similar manner, it may be shown that the four-parameter group $G_4'(Bl)$ dualistic to $G_4'(Al')$ also breaks up into ∞^1 three-parameter subgroups $G_3'(Bl)a$, each characterized by a constant value of a.

61. *No Three-Parameter Subgroups of* $G_4'(Al)$. The group $G_4'(Al)$ does not break up into subgroups of the variety $G_3'(Al)a$. Let us take two collineations $T'(ABl')$ and $T_1'(AB_1l_1')$ characterized by the same value of a and combine them. The invariant figures have in common the point A and the line l. The two transformations of the kind t along Al combine according to the law $a^{a}a^{a_1} = a^{a_2}$, so also do those of the pencil through A; for, in each case, one invariant element is common. But the two transformations of the kind t', one along Al' and the other along Al_1', are so situated that we cannot assert that $\alpha_2 = \alpha + \alpha_1$. The same is true of the two transformations of the pencils through B and B_1; they are not in position to combine according to the law $\beta_2 = \beta + \beta_1$. Hence, subgroups of $G_4'(Al)$ of the variety $G_3'(Al)a$ do not exist.

THEOREM 34.—*There are five varieties of groups of collineations of type II in which all the collineations of a group are characterized by the same value of* a. *These groups are as follows:* $G_1'(ABl')a$, $\left\{\begin{matrix} G_2'(AB)a \\ G_2'(ll')a \end{matrix}\right\}$, $\left\{\begin{matrix} G_3'(Al')a \\ G_3'(Bl)a \end{matrix}\right\}$.

§4.—COLLINEATIONS OF TYPE III.

62. *Properties of a Collineation* T''. A collineation of type III is obtained as the resultant of two elations S' and S_1' whose invariant figures (Al) and (A_1l_1) are so situated that l_1 passes through A. Let the distance AA_1 be a and let the angle lAl_1 be θ; also, let the constants of S' and S_1' be α and α_1.

In order to determine the character of the resultant of S' and S_1', we must examine the resultants of the one-dimensional transformations along l_1 and through A. Along l_1, S' produces a one-dimensional transformation of the kind t', whose constant is $\alpha \sin\theta$ and S_1' an identical transformation; hence, the resultant along l_1 is t'. Through A, S' produces an identical transformation and S_1' a transformation of the kind t_1' whose constant is $a\alpha_1$; hence, the resultant through A is t_1'. Therefore, the resultant of S' and S_1' leaves invariant the point A and the line l_1 and produces along l_1 and through A one-dimensional transformations of the kind t' whose constants are respectively $\alpha \sin\theta$ and $a\alpha_1$. This

collineation of type III will be designated by T'', its invariant figure by (Al) and its constants by β and $a\alpha$.

63. *The Group $G_3''(Al)$.* The groups $H_2'(A)$ and $H_2'(l)$, where A lies on l, each contain ∞^2 elations, and hence there are ∞^4 pairs of elations, one from each group, whose resultant in each case is a collineation of type III and whose invariant figure is the lineal element Al. It does not follow that the ∞^4 collineations T'' thus produced are all different; in fact, there are only ∞^3 different collineations T'' thus produced.

These ∞^3 collineations T'', each leaving Al invariant, form a three-parameter group, as may be shown. Let T'' and T_1'' be two collineations from the system leaving Al invariant. Along l the resultant of the two transformations of the kind t' is another of the same kind, since they have the invariant point A in common; through A the resultant of the two transformations of the kind t' is also another of the same kind, since they have the invariant ray l in common. Hence, the resultant of T'' and T_1'' is T_2', a collineation of type III, leaving Al invariant. Therefore, all collineations of type III, leaving Al invariant, form a three-parameter group $G_3''(Al)$. Evidently, $G_3''(Al)$ contains $H_2'(A)$ and $H_2'(l)$ as subgroups.

THEOREM 35.—*The aggregate of all collineations of type III, leaving invariant the same lineal element Al, forms a three-parameter group $G_3''(Al)$.*

64. *The Group $G_2''(Al)$.* In article 43, it was pointed out that the system of ∞^3 of conics, touching l at A, is made up of ∞^1 nets N of ∞^2 conics each; all the conics of each net have contact of the second order at A. (All the conics of the net N have the same circle of curvature at A.) Such a net of conics is transformed by any collineation into a net of the same kind. Hence, a collineation T'' of the group $G_3''(Al)$ transforms a net N of the system into another net N_1 of the same system or into itself. Since $G_3''(Al)$ contains ∞^3 collineations T'', and there are only ∞^1 nets N in the system, it follows that each net N is transformed into itself by ∞^2 collineations T'' belonging to the group $G_3''(Al)$. All collineations T'' in $G_3''(Al)$, which leave invariant a net of conics N, form a two-parameter subgroup $G_2''(Al)$. Evidently, $G_3''(Al)$ contains ∞^1 such subgroups, one for each net N in the system of conics touching l at A.

THEOREM 36.—*The aggregate of all collincations of type III, leaving invariant the lineal element Al and a net of conics having contact of the second order at A, forms a two-parameter group $G_2''(Al)$.*

65. *The Group $G_1''(Al)$.* The system of ∞^3 conics touching l at A is composed of ∞^2 pencils of conics, each pencil being composed of ∞^1 conics having third order contact at A. Such a pencil of conics S is transformed by any collineation into another pencil of the same kind. Since there are ∞^2 such pencils S in the system of conics touching l at A and ∞^3 collineations T'' in the group $G_3''(Al)$, it follows that there are ∞^1 collineations T'' in $G_3''(Al)$ that transform any given pencil S into itself. These collineations form a one-parameter subgroup of $G_3''(Al)$. Evidently, $G_3''(Al)$ contains ∞^2 such subgroups, one for each pencil S in the system of conics touching l at A. Each net N of the system contains ∞^1 pencils S, and hence each two-parameter subgroup of $G_3''(Al)$ contains ∞^1 one parameter subgroups $G_1''(Al)$.

The conics of the pencil S are the path-curves of the group $G_1''(Al)$; for, if all the conics of S remain invariant under the collineation T'', the effect of T'' is to move each point of the plane along its path-curve. The parameter of the group $G_1''(Al)$ is α and the law of combination of the α's is $\alpha + \alpha_1 = \alpha_2$. Hence, the structure of the group $G_1''(Al)$ is the same as that of the group $H_1'(Al)$.

THEOREM 37.—*The aggregate of all collineations of type III, leaving invariant the lineal element Al and a pencil of conics having contact of the third order at A, forms a one-parameter group $G_1''(Al)$, whose path-curves are the conics of the pencil.*

These three are the only varieties of groups of type III. It may readily be verified that two collineations of this type, with their invariant figures in relative positions other than those discussed above, result in a collineation not of type III.

The following facts regarding the number of collineations of the different types should be noted here:

There are	∞^8	collineations	of	type	I	which	form	a	group	G_8.
"	"	∞^7	"	"	II	"	do	not	form	a group.
"	"	∞^6	"	"	III	"	"	"	"	"
"	"	∞^5	"	"	IV	"	"	"	"	"
"	"	∞^4	"	"	V	"	"	"	"	"

§5. GROUP STRUCTURE AND SINGULAR TRANSFORMATIONS.

A. Structure of the Collineation Groups of the Plane.

66. Having found a complete list of the Lie groups of collineations in the plane, we must now examine more closely into the structure of each group.

Many of these groups will be found to contain collineations of one type only, while others will be found to contain collineations of two or more types. In every case, there are collineations of a characteristic type which make up all or the greater part of the group, and among these are to be found in many cases a smaller number of collineations of one or more lower types. In most instances, the collineations of these secondary types form continuous subgroups of the given group; but sometimes these secondary collineations in a given group do not form a continuous subgroup, in which case they are called *Singular Transformations.*

The complete structure of some of these collineation groups has already been given while in other cases only the characteristic collineations of the group have been indicated. The entire list of these groups must be examined; in most cases the structure of a group will be given without proof, only in those cases where the structure is not at once evident, or where the group contains singular transformations, will the proofs be indicated. Perspective collineations constitute type IV, and elations type V.

67. *Structure of the Perspective Groups.* We found in article 7 eight varieties of perspective groups, viz. three of type V and five of type IV. The group $H_1'(Al)$ contains only collineations of type V, and $H_1(A, l)$, only those of type IV. The structure of the other six perspective groups is here indicated by symbolic equations, see article 19:

$$\begin{aligned}
H_2'(l) &= \infty^1 H_1'(\overline{A}\, l),\\
H_2'(A) &= \infty^1 H_1'(A\, \overline{l}),\\
H_2(lBl') &= \infty^1 H_1(\overline{A}, l) + H_1'(Bl),\\
H_2(Al'B) &= \infty^1 H_1(A, \overline{l}) + H_1'(Al'),\\
H_3(l) &= \infty^2 H_1(\overline{A}, l) + H_2'(l),\\
H_3(A) &= \infty^2 H_1(A, \overline{l}) + H_2'(A).
\end{aligned}$$

The correctness of each of these structural formulæ may be proved in detail; but it is necessary to carry out the proof in only one case. Take, for example, the group $H_2(lBl')$; we can show that this group must contain collineations of type V. Take from the group $H_2(lBl')$ two perspective collineations $S(A, l)$ and $S_1(A_1, l)$, for which the cross-ratio constants are k and $1/k$ respectively. Along the line A_1AB, where B is the intersection of l and l', we have two one-dimensional transformations of the kind t with reciprocal values of k but belonging to different subgroups having one invariant point B in common. Hence, their resul-

tant along l' is of the kind l', and the value of α for this resultant is found to be

$$\alpha = (1-k)\left(\frac{1}{A_1} - \frac{1}{A}\right).$$

Since both transformations along l are identical, their resultant is also identical. Hence, the resultant of S and S_1 is an elation S', which belongs to the group $H_1'(Bl)$. From the above value of α we see that by varying the value of k, or the position of the points A and A_1, all elations in the group $H_1'(Bl)$ are obtained; thus, the group $H_2(lBl')$ contains $H_1'(Bl)$ as a subgroup. In a similar manner the structural formulæ of the other perspective groups may be verified.

68. *Structure of Groups of Type III.* We found only three varieties of groups of type III, viz. $G_1''(Al)$, $G_2''(Al)$, $G_3''(Al)$, and these cases are easily disposed of. The group $G_1''(Al)$ contains only collineations of type III. The group $G_3''(Al)$ contains, as we saw, the groups $H_2'(A)$ and $H_2'(l)$. It may be proved without difficulty that the group $G_2''(Al)$ contains only collineations of type III. The structural formulæ of $G_2''(Al)$ and $G_3''(Al)$ are exhibited thus:

$$G_2''(Al) = \infty^1\, G_1''(Al).$$
$$G_3''(Al) = \infty^2\, G_1''(Al) + H_2'(A) + H_2'(l).$$

69. *Structure of Groups of Type II. First class.* There are six varieties of groups of this class; we have found, article 51, that the group $G_2'(ABl')$ contains the groups $H_1(B, l')$ and $H_1'(Al)$ as subgroups; and hence we must expect that subgroups of types IV and V will appear in the groups of higher orders of this class. The structural formulæ of these groups are as follows:

$$\begin{aligned}
G_2'(ABl') &= G_2'(ABl') + H_1(B, l') + H_1'(Al),\\
G_3'(AB) &= \infty^1 G_2'(AB\bar{l}') + H_2(BlA) + H_2'(l),\\
G_3'(ll') &= \infty^1 G_2'(A\bar{B}l') + H_2(lAl') + H_2'(A),\\
G_4'(Al') &= \infty^2 G_2'(A\bar{B}\bar{l}') + H_3(l') + H_2'(A) + H_2'(l') + G_3''(Al'),\\
G_4'(Bl) &= \infty^2 G_2'(\bar{A}B\bar{l}') + (H_3 B) + H_2'(l) + H_2'(B) + G_3''(Bl),\\
G_4'(Al) &= \infty^2 G_2'(A\bar{B}\bar{l}') + \infty^2 H_1(\bar{B}, \bar{l}') + H_2'(A) + H_2'(l) + G_3''(Al).
\end{aligned}$$

In the case of the group $G_3'(AB)$, we may verify the structural formula as follows: The line l' through A may take ∞^1 different positions, and for each position of l' we have a two-parameter group $G_2'(ABl')$, all of which are contained in $G_3'(AB)$. The ∞^1 one-parameter groups $H_1(B, \bar{l}')$ in these groups

$G_2'(AB\bar{l}')$, form the two-parameter group $H_2(BlA)$. The group $H_2(BlA)$ contains the subgroup $H_1'(Bl)$, and each group $G_2'(AB\bar{l}')$ contains the subgroup $H_1'(Al)$. By combining the collineations of these two groups, viz. $H_1'(Al)$ and $H_1'(Bl)$, we obtain all the elations of the group $H_2'(l)$, which is, therefore, contained in the group $G_3'(AB)$.

As another example, let us examine the structure of the group $G_4'(A\bar{l}')$. The point B may take ∞^2 different positions in the plane, and for each position of B there is a group $G_2'(A\bar{B}\bar{l}')$ which belongs to the group $G_4'(A\bar{l}')$. The ∞^2 one-parameter perspective groups $H_1(\bar{B}, \bar{l}')$, contained in these groups $G_2'(AB\bar{l}')$, form the group $H_3(\bar{l}')$. The group $H_3(\bar{l}')$ contains the subgroup $H_2'(\bar{l}')$. The ∞^1 groups of elations $H_1'(A\bar{l})$, contained in the groups $G_2'(A\bar{B}\bar{l}')$, form the group $H_2'(A)$, which is, therefore, contained in $G_4'(A\bar{l}')$. Since $G_4'(A\bar{l}')$ contains $H_2'(A)$ and $H_2'(l)$, it must also contain $G_3''(Al)$. Thus the structure of $G_4'(A\bar{l}')$ is found. A similar course of reasoning leads to the structural formulæ of the other groups of this class.

70. *Second Class.* The group $G_1'(AB\bar{l}')a$, where $a \neq 0$ or ∞, contains only collineations of type II. The structure of the other four groups of this class is shown as follows:

$$
\begin{aligned}
G_2'(AB)a &= \infty^1 G_1'(A\bar{B}\bar{l}')a + H_1'(Bl) + \text{S. T.},\\
G_2'(l\bar{l}')a &= \infty^1 G_1'(A\bar{B}\bar{l}')a + H_1'(A\bar{l}') + \text{S. T.},\\
G_3'(A\bar{l}')a &= \infty^2 G_1'(A\bar{B}\bar{l}')a + H_2'(\bar{l}') + \text{S. T.},\\
G_3'(Bl)a &= \infty^2 G_1'(\bar{A}Bl)a + H_2'(B) + \text{S. T.}
\end{aligned}
$$

In this class of groups we meet, for the first time, with so-called singular transformations S. T.; these will be discussed later.

In order to show that the group of elations $H_1'(Bl)$ occurs in $G_2'(AB)a$, we take two collineations $T'(AB\bar{l}')a$ and $T_1'(AB\bar{l}_1')a$, whose constants are (β, a^β) and $(-\beta, a^{-\beta})$ respectively, and form their resultant. The resultant of the two one-dimensional transformations through B is identical, that along l is also identical, while that through A is of the kind t'. Hence, the resultant of T' and T_1' is an elation S' belonging to the group $H_1'(Bl)$. By varying the value of β and the positions of $\bar{l}'$ and $\bar{l}_1'$, all elations in $H_1'(Bl)$ may be produced, thus showing that all elations of the group $H_1'(Bl)$ are contained in $G_2'(AB)a$.

In like manner the structural formulæ for the remaining groups of this class may be verified.

71. *Structure of Groups of Type I. First Class.* The nine groups of type I, first class, show the following structural formulæ:

$$
\begin{aligned}
G_2(ABC) &= G_2(ABC) + H_1(Al'') + H_1(Bl') + H_1(Cl),\\
G_3(AB\overline{l'}) &= \infty^1 G_2(AB\overline{C}) + G_2'(ABl') + H_2(lAl') + H_2(AlB)\\
&\qquad + H_1(B,l') + H_1'(Al),\\
G_4(AB) &= \infty^2 G_2(AB\overline{C}) + 2\infty^1 G_2'(AB\overline{l''}) + H_3(l) + H_2'(l) + 2H_2(Al'B),\\
G_4(ll') &= \infty^2 G_2(A\overline{B}\overline{C}) + 2\infty^1 G_2'(A\overline{B}l') + H_3(A) + H_2'(A) + 2H_2(lAl'),\\
G_4(A, l'') &= \infty^2 G_2(A\overline{B}\overline{C}) + \infty^1 G_2'(A\overline{B}l'') + \infty^1 H_2(\overline{B}l\overline{A})\\
&\qquad + H_1(A, l'') + \infty^1 H_1'(\overline{B}l),\\
G_5(Al) &= \infty^3 G_2(A\overline{B}\overline{C}) + \infty^2 G_2'(A\overline{B}\overline{l'}) + \infty^2 G_2'(A\overline{B}l') + G_3''(Al) + H_3(l)\\
&\qquad + H_3(A) + H_2'(l) + H_2'(A),\\
G_6(l) &= \infty^4 G_2(ABC) + \infty^3 G_2'(\overline{A}\overline{B}l') + \infty^3 G_2'(ABl) + \infty^1 G_3''(\overline{A}l)\\
&\qquad + H_3(l) + \infty^2 H_2(\overline{A}l\overline{B}) + H_2'(l) + \infty^1 H_2'(A),\\
G_6(A) &= \infty^4 G_2(ABC) + \infty^3 G_2'(ABl) + \infty^3 G_1'(ABl') + \infty^1 G_3''(A\overline{l})\\
&\qquad + H_3(A) + \infty^2 H_2(lAl') + H_2'(A) + \infty^1 H_2'(l),\\
G_8 &= \infty^6 G_2(ABC) + \infty^5 G_2'(ABl') + \infty^3 G_3''(Al) + \infty^4 H_1(A, l'') + \infty^3 H_1'(Al).
\end{aligned}
$$

The verification of these structural formulæ presents no special difficulties.

72. *Second Class.* The structural formulæ of the four following groups are exhibited thus:

$$
\begin{aligned}
G_2(ABl')r &= \infty^1 G_1(AB\overline{C})r + H_1'(Al) + \text{S. T.},\\
G_3(AB)r &= \infty^2 G_1(AB\overline{C})r + H_2'(l) + \text{S. T.},\\
G_3(ll')r &= \infty^2 G_1(A\overline{B}\overline{C})r + H_2'(A) + \text{S. T.},\\
G_4(Al)r &= \infty^3 G_1(A\overline{B}\overline{C})r + H_2'(A) + H_2'(l) + G_3''(Al) + \text{S. T.}
\end{aligned}
$$

From the group $G_2(ABl')r$ take two collineations $T(ABC)r$ and $T(ABC_1)r$, where C and C_1 are collinear with A, whose cross-ratio constants are respectively k and $1/k$. The resultants along AB and through A are identical, while those along Al' and through B are of the kind t'. The constant of the resultant along Al' is given by $\alpha = (1-k)\left(\frac{1}{AC_1} - \frac{1}{AC}\right)$. Hence, the resultant of T and T_1 is an elation and belongs to the group $H_1'(Al)$. By varying k and the positions of C and C_1, α can be made to take on all values; hence, all elations in the group $H_1'(Al)$ are contained in $G_2(ABl')r$.

Again, from the group $G_3(AB)r$, take two collineations $T(ABC)r$ and $T_1(ABC_1)r$, where C and C_1 are not collinear with A or B, and whose cross-ratio constants are k and $1/k$ respectively. The resultant along AB is identical, while those through A and B are both of the kind t'. Hence, the resultant of T and T_1 is $S'(Xl)$, where X is some point on l. It may be shown that all elations in $H_2'(l)$ are contained in $G_3(AB)r$.

In like manner, the formulæ of the other groups of this class may be verified.

73. *Third Class.* The list of groups of the third class shows structural formulæ as follows:

$$\begin{aligned}
G_2(AKl) &= \infty^1 G_1(ABC) + G_1'(Al),\\
G_3(K) &= \infty^2 G_1(ABC) + \infty^1 G_1''(Al),\\
G_3(S) &= \infty^2 G_1(ABC) + G_1''(Al) + H_1'(Al) + \text{S. T.},\\
G_3(A, l'')_{r=2} &= \infty^2 G_1(ABC) + \infty^1 H_1'(A\bar{B}\bar{l}'') + \text{S. T.},\\
G_5(l'')_{r=2} &= \infty^4 G_1(ABC) + \infty^1 H_2'(Al) + \text{S. T.},\\
G_5(A)_{r=2} &= \infty^4 G_1(ABC) + \infty^1 H_2'(l) + \text{S. T.}
\end{aligned}$$

The group $G_3(S)$ contains ∞^2 subgroups $G_1(ABC)$, where B is in turn every point on Al and C in turn every point in the plane. One conic of the system S passes through C, and BC is tangent to this conic at C. Take a collineation T from the group $G_1(ABC)$ and another T_1 from the group $G_1(AB_1C_1)$ having cross-ratio constants k and $1/k$ respectively. Their resultant is of the kind t' along Al and through A, and hence is a collineation T'' of type III belonging to the group $G_1''(Al)$. It can be shown that all the collineations of the group $G_1''(Al)$ are to be found in $G_3(S)$.

Again, take two collineations T and T_1, whose cross-ratio constants are k and $1/k$, from the groups $G_1(ABC)$ and $G_1(ABC_1)$, where C and C_1 are collinear with A. Their resultant is identical along AB and through A, but is of the kind t' along ACC_1 and through B; hence, it is an elation S' and belongs to the group $H_1'(Al)$. Evidently, all elations in $H_1'(Al)$ are contained in $G_3(S)$.

The remaining formulæ are easily verified.

*B.—Singular Transformations.**

We come next to the consideration of singular transformations in the colline-

* The results of the investigation of singular transformations by the present methods agree with those found by Professor Taber and his pupils, Rettger, Slocum, and Williams. See Taber, Bull. Amer. Math. Soc., Vol. VI, p. 199; Rettger, Amer. Jour. Math., Vol. XXII, p. 60; Williams, Proc. Amer. Acad., Vol. XXXV, p. 97; Slocum, Proc. Amer. Acad., Vol. XXXVI, p. 92.

ation groups of the plane; these were defined, article 66, as systems of collineations of one type not forming a continuous group, yet occurring in an otherwise continuous group of another type. We shall find two distinct kinds of singular transformations, viz. discrete systems of collineations of type V or III occurring in groups of type II, second class; and discrete systems of collineations of type II occurring in groups of type I, second and third class.

74. *Singular Transformations in* $G_2'(AB)a$ *and* $G_2'(ll')a$. It was shown, article 70, that the group $G_2'(AB)a$ contains $H_1'(Bl)$ as a subgroup. It also contains other collineations of type V, as will be shown. Take two collineations $T'(ABl')$ and $T_1'(ABl_1')$, whose constants are (β, a^β) and (β_1, a^{β_1}); for every value of $a \neq 0$ or ∞, β and β_1 may be so chosen that $\beta + \beta_1 \neq 0$, while $a^{\beta+\beta_1} = 1$. Put $a = re^{i\theta}$ and $\beta + \beta_1 = p + iq$, then we have

$$(p + iq)(\log r + i\theta) = 2n\pi i,$$

whence $$p \log r - q\theta = 0 \text{ and } q \log r + p\theta = 2n\pi;$$

also, $$p = \frac{2n\pi\theta}{\log^2 r + \theta^2} \quad \text{and} \quad q = \frac{2n\pi \log r}{\log^2 r + \theta^2},$$

Hence, for any given value of a, β and β_1 can always be found to satisfy the conditions

$$\beta + \beta_1 \neq 0 \quad \text{and} \quad a^{\beta+\beta_1} = 1.$$

The resultant of T' and T_1' is identical along AB, but the resultant one-dimensional transformations through A and B are both of the kind t'. Hence, the resultant is an elation $S'(Xl)$, where X is some point on AB, not A or B. In the above expressions for p and q, n has only integral values; hence, the resultant elations S' do not form a continuous system. For any value of $a \neq 0$ or ∞, $\beta + \beta_1$ cannot be infinitesimal and satisfy the above conditions so that all the elations in $H_2'(l)$ cannot be obtained in this manner. Hence, $G_2'(AB)a$ contains, besides the group $H_1'(Bl)$, a system of elations $S'(Xl)$ not forming a continuous group; these are singular transformations.

In exactly the same manner it may be shown that the group $G_2'(ll')a$ contains, besides the subgroup of elations $H_1'(Al')$, a discrete system of elations $S'(A\lambda)$ not forming a continuous group. These are, therefore, singular transformations in $G_2'(ll')a$.

THEOREM 38.—*The groups* $G_2'(AB)a$ *and* $G_2'(ll')a$ *each contain a discrete system of elations which are singular transformations of the groups.*

75. *Singular Transformations in* $G_3'(Bl)a$ *and* $G_3'(Al')a$. The group $G_3'(Bl)a$ contains ∞^1 subgroups $G_2'(AB)a$, one for each point A on the line l and the group $H_2'(B)$; it, therefore, contains all the singular transformations of type V to be found in its subgroups $G_2'(AB)a$. Take two collineations $T'(ABl')$ and $T_1'(A_1Bl_1')$ whose constants are (β, a^β) and (β_1, a^{β_1}). If β and β_1 are so chosen that $\beta + \beta_1 = 0$, the resultant of T' and T_1' is identical through B and of the kind t' along l; hence, it is an elation belonging to the group $H_2'(B)$. But if $\beta + \beta_1 \neq 0$ and $a^{\beta+\beta_1} = 1$, the resultants through B and along l are both of the kind t'; hence, it is a collineation of type III and belongs to the group $G_1''(Bl)$. For the same reasons given above, no infinitesimal collineation of type III can be produced in this manner, and hence all collineations in $G_1''(Bl)$ are not to be found in $G_3'(Bl)a$. These singular collineations of Type III in $G_3'(Bl)a$ also appear as the resultants of the singular collineations of type V in $G_2'(Bl)a$ and the elations of the group $H_2'(B)$.

In like manner it can be shown that the group $G_3'(Al')a$ contains singular transformations of types V and III.

THEOREM 39.—*The groups* $G_3'(Bl)a$ *and* $G_3'(Al')a$ *each contain discrete systems of collineations of both types III and V which are singular transformations of the groups.*

76. *Singular Transformations in* $G_2(ABl')r$. We shall next examine the group $G_2(ABl')r$ for singular transformations. There are two cases to be considered according as r is a rational or an irrational number. When r is rational, the group $G_1(ABC)r$ contains (art. 26), one perspective involutoric collineation S, whose vertex and axis are respectively a vertex and the opposite side of the invariant triangle (ABC).

Suppose that r has odd numerator and odd denominator; then, according to the notation agreed upon, S has its vertex at B and its axis along AC. For such a value of r the group $G_2(ABl')r$ contains ∞^1 subgroups $G_1(ABC)r$; but these all contain the same perspective collineation S; hence, the group $G_2(ABl')r$ contains but one perspective involutoric collineation S. Let S be combined with any collineation of type I in $G_2(ABl')r$; the resultant one-dimensional transformations along both AB and AC are of the kind t, and hence, the resultant of S and T is of type I and is one of the collineations of the group $G_2(ABl')r$. On the other hand, let S be combined with S', a collineation of type V in $H_1'(Al)$

(which is a subgroup of $G_2(AB\ell')r$). Along AB we have an involutoric combined with an identical transformation, and the resultant in this direction is involutoric with invariant points at A and B. Along AC we have an identical combined with a transformation of the kind t', and the resultant is of the kind t'. Through B we have an identical combined with a transformation of the kind t', and the resultant is also of the kind t'. Hence, the resultant of S and S' is T', a collineation of type II, whose invariant figure is $(AB\ell')$ and whose constants k and α are $k=-1$ and α equal the α of S. If S be combined in turn with each elation of $H_1'(A\ell)$, we have an infinite system of collineations of type II, for each of which $k=-1$ while α has all complex values. The parameters k and α of this system of collineations of type II do not satisfy the relation $k=\alpha^{\alpha}$, and hence (article 49) do not form a continuous group. These collineations of type II in $G_2(AB\ell')r$ are singular transformations. The structure of $G_2(AB\ell')r$ is thus seen to be

$$G_2(AB\ell')r=\infty G_1(AB\overline{C})r+H_1'(A\ell)+\infty^1 T'.$$

Let us next examine for singular transformations the group $G_2(AB\ell')r$, where r is rational with odd numerator and even denominator. Each subgroup $G_1(ABC)r$ of $G_2(AB\ell')r$ contains one perspective involutoric collineation S; these are all different, and form a system (S) whose common axis is AB and whose vertices are in turn every point on ℓ'. Any one of these perspective collineations, combined with a collineation of type I in $G_2(AB\ell')r$, results in a collineation of type I also belonging to $G_2(AB\ell')r$. The resultant of any collineation of the system (S) with any elation of the group $H_1'(A\ell)$ is another perspective collineation S_1 of the system (S). The resultant of any two perspective collineations of the system (S) is an elation belonging to the group $H_1'(A\ell)$. Thus the group $G_2(AB\ell')r$, where r is rational with odd numerator and even denominator, contains ∞^1 involutoric perspective collineations, but no singular transformations of type II. These perspective collineations are not singular transformations in the sense of the definition, for each of them belongs to a subgroup of $G_2(AB\ell')r$.

In like manner, it may be shown that the group $G_2(AB\ell')r$, where r is rational with even numerator and odd denominator, contains ∞^1 involutoric perspective collineations having a common vertex at A and axes in turn every line through B; but it contains no singular transformations. The group $G_2(AB\ell')r$,

where r is irrational, contains (article 27) no involutoric perspective collineation and hence no singular transformations.

From these results, we conclude that the group $G_2(AB\overline{l}')r$ contains singular transformations of type II when, and only when, its subgroups have one common involutoric perspective collineation S. This depends on the manner in which the invariant triangles of the subgroups $G_1(ABC)r$ are put together. Thus, when the line $\overline{l}'$ passes through A, the group $G_2(AB\overline{l}')r$ contains singular transformations when r has odd numerator and odd denominator. On the other hand, if l' passes through B, in which case it is designated by $\overline{l}''$, the group $G_2(AB\overline{l}'')r$ has singular transformations when r has even numerator and odd denominator.

THEOREM 40.—*The group $G_2(AB\overline{l}')r$, where r is a rational number such that the subgroups of $G_2(AB\overline{l}')r$ all contain the same involutoric perspective collineation S, contains singular transformations; these are of type II, and all have the same value of k, viz. $k=-1$.*

77. *Other Groups Containing Singular Transformations of Type II.* Any collineation group of the plane which contains no subgroup of type II and which contains subgroups of the variety $G_2(AB\overline{l}')r$ such that each contains but a single perspective involutoric collineation S, will evidently contain singular transformations of type II. In addition to $G_2(AB\overline{l}')r$, the following groups of type I, second class, also contain singular transformations: $G_3(AB)r$, $G_3(l\overline{l}')r$, $G_4(Al)r$.

In type I, third class, the group $G_3(S)$ contains ∞^4 subgroups $G_2(AB\overline{l}')r=-1$. Each of these subgroups contains ∞^1 singular transformations, and hence $G_3(S)$ contains ∞^2 singular collineations of type II.

The group $G_3(A, \overline{l}'')$ contains ∞^1 subgroups $G_2(AB\overline{l}'')r=2$. There is evidently but one involutoric perspective collineation S in the group $G_3(A, \overline{l}'')$, and this has its vertex at A and its axis coinciding with $\overline{l}''$; S belongs to every subgroup $G_2(AB\overline{l}'')r=2$ in $G_3(A, \overline{l}'')$. S, combined in turn with each of the ∞^2 elations in $G_3(A, \overline{l}'')$, gives ∞^2 singular transformations of type II.

The group $G_5(\overline{l}'')$ contains ∞^2 subgroups $G_3(A, \overline{l}'')$, one for each position of A in the plane; hence, it contains ∞^3 subgroups $G_2(AB\overline{l}'')r=2$. Since each of these two-parameter subgroups contains ∞^1 singular transformations of type II, it follows that the group $G_5(\overline{l}'')$ contains ∞^4 such singular transformations.

The group $G_5(A)$, dualistic to $G_5(\overline{l}'')$, also contains ∞^4 singular transformations of type II.

THEOREM 41.— *The following groups, and no others, contain singular transformations of type II:* $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$, *and* $G_4(Al)\,r$ *(when* r *is rational);* $G_3(S)$, $G_3(A, l')_{r=2}$, $G_5(l')_{r=2}$ *and* $G_5(A)_{r=2}$.

§6.—MIXED GROUPS OF COLLINEATIONS IN THE PLANE.

78. In this section the following problems will be investigated: To find (1) all collineations in the plane that leave fixed one vertex of a triangle and interchange the other two vertices; (2) that interchange a pair of points; (3) that interchange a pair of lines; (4) that permute the vertices of a triangle; (5) that permute the vertices of a quadrilateral; (6) that permute the vertices of a polygon of n sides.

The first four of these problems lead us to the consideration of certain mixed groups of collineations. For example, all collineations, leaving a triangle invariant as a whole, form a mixed group $mG(ABC)$, which consists of the continuous group $G_2(ABC)$ and all other collineations interchanging a pair of its vertices and also those permuting the three vertices.

79. *The Mixed Groups* $mG_1(PQ)$ *and* $mG_1(ll')$. As a necessary preliminary, we must first determine all one-dimensional projective transformations that interchange a pair of points on a line. There are ∞^1 projective transformations of the points on a line that leave invariant a pair of points P and Q; these form a continuous group $G_1(PQ)$. The aggregate of all transformations of both kinds, i. e. those that leave P and Q separately invariant and those that interchange P and Q, forms the mixed group $mG_1(PQ)$.

Every transformation interchanging P and Q is an involutoric transformation of the kind t, and its cross-ratio is $k=-1$. If the invariant points of t are A and B, they must be so situated that the cross-ratio $(ABPQ)=-1$. There are ∞^1 pairs of points AB having this relation; hence, there are ∞^1 involutoric transformations t interchanging P and Q. Since no transformation of the kind t' can be involutoric, it follows that there are no transformations of the kind t' interchanging P and Q,

It is evident that similar results hold also for the mixed group of one-dimensional transformations, leaving invariant and interchanging a pair of lines.

THEOREM 42.—*The mixed group* $\left\{\begin{matrix} mG_1(PQ) \\ mG_1(ll') \end{matrix}\right\}$ *contains* ∞^1 *involutoric trans-*

formations in addition to the transformations of the continuous group $\begin{Bmatrix} G_1(PQ) \\ G_1(ll') \end{Bmatrix}$; *it contains no other transformations.*

80. *The Mixed Group* $mG_2(APQ)$. Let (APQ) be any three points forming a triangle. Any collineation that interchanges P and Q must leave the line PQ invariant. Let (ABC) be the invariant triangle of a collineation of type I which interchanges P and Q; then two of the vertices, as B and C, must be on the line PQ and so situated that the cross-ratio $(BCPQ) = -1$. All collineations in the group $G_2(ABC)$, for which the cross-ratio along the side BC is -1, leave A invariant and interchange P and Q; they belong, therefore, to the mixed group $mG_2(APQ)$.

Let k and k' be the two independent parameters of the continuous group $G_2(ABC)$, and let k be taken along AB and k' along AC. Since the product of the three cross-ratios taken in the same order around the triangle must be unity, we have $(k)(-1)(1/k') = 1$; thus, $k + k' = 0$. Hence, out of the ∞^2 collineations in $G_2(ABC)$, where $(BCPQ) = -1$, there are ∞^1 that satisfy the condition $k + k' = 0$ and interchange P and Q. The pair of points BC can be chosen in ∞^1 different ways, so that $(BCPQ) = -1$; if out of each of these groups $G_2(ABC)$ we select the collineations that satisfy the relation $k + k' = 0$, we obtain ∞^2 collineations of type I that leave A invariant and interchange P and Q.

Among these ∞^2 collineations of type I interchanging P and Q, there are ∞^1 of type IV. In every group $G_2(ABC)$, there are two collineations, viz. $k = 1$, $k' = -1$ and $k = -1$, $k' = 1$ which satisfy the condition $k + k' = 0$ and are not of type I. They are involutoric perspective collineations with the vertex always on the line PQ.

Since no collineation of types III or V is involutoric along an invariant line, it follows that the mixed group $mG_2(APQ)$ contains no collineations of these types. A collineation of type II, whose invariant figure is (BCl'), may be involutoric along BC, but it cannot belong to the mixed group $mG_2(APQ)$, for the point A would then have to be a second invariant point on the line l', which is impossible.

THEOREM 43.—*The mixed group* $mG_2(APQ)$ *contains, in addition to the continuous group* $G_2(APQ)$, ∞^2 *collineations of type I and* ∞^1 *of type IV, which leave A invariant and interchange P and Q.*

81. *The Mixed Groups* $mG_4(PQ)$ *and* $mG_4(ll')$. The continuous group $G_4(PQ)$ contains ∞^4 collineations, leaving P and Q separately invariant; we seek, in addition to these, all collineations which interchange P and Q. There are ∞^3 triangles (ABC) so situated that $BCPQ$ are collinear and the cross-ratio $(BCPQ) = -1$. Each of these triangles is the invariant triangles of a two-parameter group in which ∞^1 collineations satisfy the relation $k + k' = 0$, and hence interchange P and Q. Therefore, there are ∞^4 collineations of type I which interchange P and Q.

Let us consider the group $G_2'(BCl')$ of type II, where $(BCPQ) = -1$, The collineations of this group depend upon two-parameters k and α. When $k = -1$, the transformations along BC are involutoric and interchange P and Q. The group $G_2'(BCl')$ contains ∞^1 such collineations, one for each value of α. The figure (BCl') can be chosen in ∞^2 different positions satisfying the condition $(BCPQ) = -1$. Hence, there are ∞^3 collineations of type II which interchange P and Q.

Let g be any line of the plane cutting PQ in B and take C on PQ such that $(BCPQ) = -1$. The involutoric collineation of the group $H_1(C, g)$ interchanges P and Q. There are evidently ∞^2 such involutoric collineations, one for each line of the plane not passing through P or Q.

In like manner, it may be shown that the mixed group $mG_4(ll')$ has a similar structure to $mG_4(PQ)$; these groups are dualistic, and the properties of the former may be inferred at once from those of the latter.

THEOREM 44.—*The mixed group* $\left\{\begin{matrix} mG_4(PQ) \\ mG_4(ll') \end{matrix}\right\}$ *contains, besides the continuous group* $\left\{\begin{matrix} G_4(PQ) \\ G_4(ll') \end{matrix}\right\}$, ∞^4 *collineations of type I,* ∞^3 *of type II and* ∞^2 *of type IV, which interchange* $\left\{\begin{matrix} P \\ l \end{matrix}\right\}$ *and* $\left\{\begin{matrix} Q \\ l' \end{matrix}\right\}$.

82. *Collineations which Permute the Vertices of a Triangle.* Let P, Q, R be the vertices of a triangle; we wish to find all collineations which change P into Q, Q into R and R into P; also, their inverses, viz. those that change P into R, R into Q and Q into P. Let K be any conic circumscribing the triangle; with PQR as the triangle of reference, the homogeneous equation of K may be written

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0. \tag{1}$$

Let x', y', z' be the coordinates of any point A; the polar of A with respect to the triangle of reference is

$$\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0. \tag{2}$$

The polar of A with respect to the conic K is given by

$$x(bz' + cy') + y(az' + cx') + z(ay' + bx') = 0. \tag{3}$$

We wish to determine the point x', y', z', so that its polars with respect to the triangle (PQR) and the conic K coincide. Comparing equations (2) and (3), we find $x' : y' : z' = a : b : c$.

When the point A is not on a side of the triangle of reference, we find one, and only one, position of A such that its polar with respect to the triangle is at the same time its polar with respect to the conic K. The converse of this proposition is also true; if we choose any point A and take its polar l with respect to the triangle of reference, we shall find one, and only one, conic K circumscribing the triangle for which A and l are pole and polar.

The line l and conic K, whose equations are respectively

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \quad \text{and} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

intersect in a pair of points B and C. (ABC) is the invariant triangle of a two-parameter group $G_2(ABC)$; the path-curves of its one-parameter subgroup $G_1(ABC)\, r = 2$ are conics and K is one of these path-curves. Consequently, all collineations of the group $G_1(ABC)\, r = 2$ transform K into itself.

The coordinates of A are (a, b, c) of $B(a, \omega b, \omega^2 c)$, of $C(a, \omega^2 b, \omega c)$, where $\omega^3 = 1$. The lines AB, AC, AP, AQ, AR are given by the following equations respectively:

$$\frac{x}{a} + \frac{y}{\omega b} + \frac{z}{\omega^2 c} = 0, \quad \frac{x}{a} + \frac{y}{\omega^2 b} + \frac{z}{\omega c} = 0,$$

$$\frac{y}{b} = \frac{z}{c}, \quad \frac{x}{a} = \frac{z}{c}, \quad \frac{x}{a} = \frac{y}{b}, \tag{4}$$

From these equations we readily find that the cross-ratios of the three pencils $A(BCPQ)$ $A(BCQR)$, $A(BCRP)$ are all equal to each other and each equal to ω, i. e. $e^{\frac{2\pi i}{3}}$. Consequently, the collineations of the group $G_1(ABC)$ $r=2$, for which $k = e^{\frac{2\pi i}{3}}$ changes P into Q, Q into R, and R into P. The inverse of this collineation, for which $k = e^{\frac{4\pi i}{3}}$, changes P into R, R into Q, and Q into P.

There are ∞^2 conics circumscribing the triangle (PQR); for each of these conics there is a point A and a line l which are pole and polar with respect to both triangle and conic. Consequently, there are ∞^2 two-parameter groups $G_2(ABC)$, each of which contains a pair of inverse collineations that permute the vertices of the triangle (PQR). Each of these collineations is of order 3.

THEOREM 45.— *There are ∞^2 collineations of type I and period 3 that permute the vertices of a triangle.*

83. *Collineations which Permute the Vertices of a Quadrangle.* Let $(PQRS)$ be four points in a plane forming the vertices of a quadrangle; let PR and QS meet in A, PQ and RS in A'', and PS and QR in A'. Let PR and QS meet $A'A''$ in U and V respectively. On the line $A'A''$ the two pairs of points $A'A''$ and UV determine an involution whose double points are designated by B and C. The two-parameter group $G_2(ABC)$ contains three collineations which permute the vertices of the quadrangle $(PQRS)$.

In order to verify this statement, we reason thus: Let T be a collineation permuting $PQRS$ in the order named; we then have the substitution T,

$$T: \left\{ \begin{matrix} P & Q & R & S \\ Q & R & S & P \end{matrix} \right\}.$$

Since the lines PQ and RS are changed into QR and SP respectively, A'', the point of intersection of the first pair, is transformed into A' the point of intersection of the second pair. In like manner, it is shown that A' is transformed into A''. Thus the line $A'A''$ is an invariant line of T, since a pair of points on it are interchanged. It may be shown, in the same way, that the lines PR and QS are interchanged; hence, their point of intersection A is an invariant point of T. Since PR and QS are interchanged while $A'A''$ remains invariant, it follows that U and V must be interchanged. Hence, the invariant triangle of T is (ABC) and the transformation is involutoric along BC.

From the above considerations, we see that T must belong to the group $G_2(ABC)$ and be one of the collineations of that group satisfying the condition $k + k' = 0$. It must also be of period 4; hence, $k^4 = 1$, i. e. $k = i, -i, -1, 1$. Taking for T that collineation for which $k = i$, its inverse is T^3 for which $k = -i$. The collineation T^2 is involutoric perspective and leaves invariant A and every point on $A'A''$.

There are two other triangles which bear to the quadrangle $(PQRS)$ the same relation as the triangle (ABC). On the line AA'', the two pairs of points

AA'' and the intersection of AA'' with PS and QR determine an involution whose double points may be designated as B' and C'. The two-parameter group $G_2(A'B'C')$ contains three collineations which also permute the vertices of the quadrangle $(PQRS)$; also for these $k=i, -i, -1, 1$. Another group $G_2(A''B''C'')$, where B'' and C'' are the double points of the involution on the line AA' determined by the pairs of points AA' and the intersection of AA' with PQ and RS, contains three collineations which also permute the vertices of the quadrangle $(PQRS)$.

THEOREM 46.—*There are (including the identical collineation) ten collineations in the plane permuting the vertices of a quadrangle* $(PQRS)$.

84. *Group of Invariant Pencil of Conics.* We are now in position to enumerate all the collineations that transform into itself a pencil of conics through four points. All collineations, leaving invariant the set of four points $(PQRS)$, also leaves invariant as a whole the pencil of conics through these points. There are two collineations of type I and period three that leave invariant one of the points as S and permute the other three; hence, there are in all eight of this character. There are also six collineations of type I and period four that permute the four points. In addition to these there are nine involutoric perspective collineations, i. e. of type IV and period two, which interchange the points in pairs, and the identical collineation. Thus there are, all told, twenty-four collineations which transform into itself a pencil of conics through four points. These form a discontinuous group of order twenty-four $G_{24}(PQRS)$. This group is abstractly identical with the well-known substitution group on four letters.

In a similar manner, it may be shown that there is a dualistic group of order twenty-four which leaves invariant a range of conics touching four lines $(pqrs)$.

THEOREM 47.—*There are twenty-four collineations of the plane which leave invariant a* $\left\{\begin{matrix}\text{pencil}\\\text{range}\end{matrix}\right\}$ *of conics* $\left\{\begin{matrix}\text{through}\\\text{touching}\end{matrix}\right\}$ *four* $\left\{\begin{matrix}\text{points}\\\text{lines}\end{matrix}\right\}$, *these form a group of order twenty-four,* $\left\{\begin{matrix}G_{24}(PQRS)\\G_{24}(pqrs)\end{matrix}\right\}$.

§7.—TABLE OF GROUPS OF COLLINEATIONS OF THE PLANE.

In this table, the collineation groups of the plane are classified according to the five types of collineations. Each group is designated by an appropriate sym-

bol. The self-dualistic groups are enclosed in brackets, thus: $\{G_2(ABC)\}$; a pair of dualistic groups are bracketed together, thus: $\left\{\begin{matrix} G_6(A) \\ G_6(l) \end{matrix}\right\}$. Similar tables are given by Lie, "Continuierliche Gruppen," pp. 288–291, and by Meyer, "Chicago Congress Papers," pp. 188–190; but in these tables of Lie and Meyer the notation and classification is entirely different. The numbers on the right refer to Lie's and Meyer's tables respectively.

TYPE I.

A.—Groups of the First Class.

	Symbol.	Invariant Figure.	Lie.	Meyer.
(1).	$\{G_2(ABC)\}$.	Triangle	(31)	(23)
(2).	$\{G_3(ABl')\}$.	Two points, their joining line, and a line through one of these points	(20)	(15)
(3).	$\left\{\begin{matrix} G_4(AB) \\ \end{matrix}\right\}$.	Two points and their joining line	(11)	(9a)
(4).	$\left\{\begin{matrix} G_4(ll') \\ \end{matrix}\right\}$.	Two lines and their point of intersection,	(12)	(9b)
(5).	$\{G_4(A, l')\}$.	Point and line, separate	(10)	(8)
(6).	$\{G_5(Al)\}$.	Lineal element	(6)	(4)
(7).	$\left\{\begin{matrix} G_6(A) \\ \end{matrix}\right\}$.	Point	(3)	(2b)
(8).	$\left\{\begin{matrix} G_6(l) \\ \end{matrix}\right\}$.	Line	(2)	(2a)
(9).	$\{G_8\}$.	No invariant figure	(1)	(1)

B.—Groups of the Second Class.

(1).	$\{G_1(ABC)r\}$.	Triangle and pencil of path-curves	(35)	(26)
(2).	$\{G_2(ABl')r\}$.	Same as $G_3(ABl')$	(27)	(20)
(3).	$\left\{\begin{matrix} G_3(AB)r \\ \end{matrix}\right\}$.	Same as $G_4(AB)$	(18)	(14a)
(4).	$\left\{\begin{matrix} G_3(ll')\ r \\ \end{matrix}\right\}$.	Same as $G_4(ll')$	(19)	(14b)
(5).	$\{G_4(Al)r\}$.	Same as $G_5(Al)$	(7)	(5)

C.—Groups of the Third Class.

(1).	$\{G_1(K_n)$.	Triangle and pencil of conics having double contact	(35)	(26)
(2).	$\{G_2(AKl)\}$.	Conic and point A on it and tangent at A,	(34)	(25)
(3).	$\{G_3(K)\}$.	Conic	(23)	(17)

		Invariant Figure.	Lie.	Meyer.
(4).	$\{G_3(S)\}$.	Lineal element Al and pencil of conics having third order contact at A and touching l	(14)	(11)
(5).	$\{G_3(A, l'')r=2\}$.	Same as $G_4(A, l'')$	(17)	(13)
(6).	$\{G_4(Al)r=-1\}$.	Lineal element	(8)	(6)
(7).	$\left\{\begin{matrix}G_5(A)r=2\\ G_5(l)r=2\end{matrix}\right\}$.	Point	(5)	(3b)
(8).		Line.....	(4)	(3a)

TYPE II.

A.—Groups of the First Class.

(1).	$\{G_2'(ABl')\}$.	Same as $G_3(ABl')$	(28)	(21)
(2).	$\left\{\begin{matrix}G_3'(AB)\\ G_3'(ll')\end{matrix}\right\}$.	Same as $G_4(AB)$	(18)	(14a)
(3).		Same as $G_4(ll')$	(19)	(14b)
(4).	$\left\{\begin{matrix}G_4'(Al')\\ G_4'(Bl)\end{matrix}\right\}$.	Lineal element	(7)	(5a)
(5).		" "	(7)	(5b)
(6).	$\{G_4'(Al)\}$.	" "	(9)	(7)

B.—Groups of the Second Class.

(1).	$\{G_1'(ABl')a\}$.	Same as $G_3(ABl')$ and pencil of path-curves	(36)	(27)
(2).	$\left\{\begin{matrix}G_2'(AB)a\\ G_2'(ll')a\end{matrix}\right\}$.	Same as $G_4(AB)$	(25)	(19a)
(3).		Same as $G_4(ll')$	(26)	(19b)
(4).	$\left\{\begin{matrix}G_3'(Al')a\\ G_3'(Bl)a\end{matrix}\right\}$.	Lineal element	(15)	(12a)
(5).		" "	(16)	(12b)

TYPE III.

(1).	$\{G_1''(Al)\}$.	Lineal element Al and pencil of ∞^1 conics having contact of third order with l at A	(37)	(28)
(2).	$\{G_2''(Al)\}$.	Lineal element Al and net of ∞^2 conics having contact of second order with l at A	(24)	(18)
(3).	$\{G_3''(Al)\}$.	Lineal element Al and ∞^3 conics touching l at A.....	(13)	(10)

TYPE IV.

		Invariant Figure.	Lie.	Meyer.
(1).	$\{H_1(A, l)\}$.	All points on l and all lines through A ..	(38)	(29)
(2).	$\left\{H_2(lBl')\right\}$.	The lines l and l' and all points on l	(32)	(24a)
(3).	$\left\{H_2(Al'B)\right\}$.	The points A and B and all lines through B	(33)	(24b)
(4).	$\left\{H_3(A)\right\}$.	All lines through A	(22)	(16b)
(5).	$\left\{H_3(l)\right\}$.	All points on l	(21)	(16a)

TYPE V.

(1).	$\{H_1'(Al)\}$.	All points on l and all lines through A ..	(39)	(30)
(2).	$\left\{H_2'(A)\right\}$.	All lines through A	(30)	(22b)
(3).	$\left\{H_2'(l)\right\}$.	All points on l	(29)	(22a)

A comparison of the tables of Lie and Meyer show them to be practically identical; in fact, Meyer has only put Lie's infinitesimal notation into finite form. A comparison of the present table with Lie's table shows some results worthy of notice. The most important result reached here is that every group is assigned to its proper type and class. This is a distinct advance upon Lie's treatment of the projective group.

The groups numbered (18) and (19) in Lie's table are the groups $G_3(AB)r$ and $G_3(ll')r$ for all values of r. These are, in general, of type I, second class, but they include also the groups $G_3'(AB)$ and $G_3'(ll')$ of type II, first class. The existence of these latter groups would hardly be suspected from Lie's or Meyer's formulæ.

Again, Lie's group (7) is, in general, $G_4(Al)r$ for all values of r, but it also includes the groups $G_4'(Al')$ and $G_4'(Bl)$. The existence of the latter as distinct groups is unknown to Lie's theory.

The group $G_4(Al)r=-1$, (8) of Lie's table, is only a special case of $G_4(Al)r$, (7) of the same table; and, though its structure is somewhat peculiar, it is doubtful if it is worthy of special mention in the list.

LAWRENCE, KANSAS, *May* 9, 1901.

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Infinitesimal Deformation of Surfaces.

By L. P. Eisenhart.

I.

Consider a flexible and inextensible surface S referred to a general system of parametric lines, u and v, and upon it a point M whose cartesian coordinates are x, y, z. Effect upon S a deformation which changes it into a surface S' and in such a way that M goes into a point M' whose coordinates are given by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \varepsilon z_1,$$

where x_1, y_1, z_1 are functions of u and v and ε is a small constant whose powers higher than the first are neglected. Since S is inextensible, $ds' = ds$, so that the functions x_1, y_1, z_1 must satisfy the condition

$$dxdx_1 + dydy_1 + dzdz_1 = 0. \qquad (\alpha)$$

Hence the effecting of the required deformation depends upon the integration of this total differential equation, for as soon as a solution is obtained an infinitesimal deformation can be made, in directions whose cosines are proportional to x_1, y_1, z_1.

As Moutard has pointed out, the functions x_1, y_1, z_1 may be looked upon as the cartesian coordinates of a surface S_1, and in consequence of condition (α), S_1 and S correspond by orthogonality of linear elements.

In section II, we determine the coefficients of the two fundamental quadratic forms of S_1 as functions of the coefficients of the quadratic forms of S and the function, to be defined later, which determines the character of the given deformation. These enable us to determine the correspondence between the conjugate and asymptotic lines on the two surfaces, and the relative character of the total curvature of these surfaces at corresponding points. We also find the relations between the Christoffel symbols formed with respect to the square of linear element of S and S_1, and from these learn the properties of the partial

differential equations, which the cartesian coordinates of a point on these surfaces satisfy when the latter are referred to particular systems of corresponding lines. And the special case is considered when the tangent planes to S and S_1 are parallel.

In section III, the surface S', which is the result of the deformation of S is studied by means of the coefficients of its two fundamental forms which are calculated as functions of the coefficients of the fundamental forms of S and S_1, and the characteristic function of the deformation. From the coefficients of the first form we pass to the Christoffel symbols, formed with respect to this form, and find the differential equations which the coordinates of S' satisfy, and by means of the coefficients of the second form, we are enabled to discover the correspondence between conjugate lines and asymptotic lines on S, S_1 and S'. Finally, we find the necessary and sufficient condition that S has been displaced merely and not deformed.

In the last section, several kinds of infinitesimal deformation of a minimal surface are treated, and a relation, from the point of view of infinitesimal deformation, is found to exist between a minimal surface, its associates and adjunct. After the consideration of the deformation of a sphere, the section closes with a discussion of two particular kinds of deformation: 1° where S and S_1 are deformed into S' and S_1', the latter corresponding by orthogonality of linear elements; 2° the case where S and S_o, the latter being a surface corresponding to S by parallelism of tangent planes, are so deformed that the resulting surfaces have the same kind of correspondence.

II

Throughout this discussion we shall use the following notation:

$$\begin{aligned} E &= \sum\left(\frac{\partial x}{\partial u}\right)^2, \quad F = \sum \frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}, \quad G = \sum\left(\frac{\partial x}{\partial v}\right)^2; \\ D &= \sum X\frac{\partial^2 x}{\partial u^2}, \quad D' = \sum X\frac{\partial^2 x}{\partial u\,\partial v}, \quad D'' = \sum X\frac{\partial^2 x}{\partial v^2}; \\ \mathfrak{E} &= \sum\left(\frac{\partial X}{\partial u}\right)^2, \quad \mathfrak{F} = \sum \frac{\partial X}{\partial u}\,\frac{\partial X}{\partial v}, \quad \mathfrak{G} = \sum\left(\frac{\partial X}{\partial v}\right)^2, \end{aligned} \tag{A}$$

where X, Y, Z are the direction cosines of the normal to the surface. If ds and $d\sigma$ denote the elements of arc on the surface S and its spherical representa-

tion respectively, then

$$\begin{aligned} ds^2 &= E du^2 + 2F du\, dv + G dv^2, \\ d\sigma^2 &= \mathfrak{E} du^2 + 2\mathfrak{F} du\, dv + \mathfrak{G} dv^2, \end{aligned} \tag{B}$$

And for brevity we put

$$H = \sqrt{EG - F^2}, \quad \mathfrak{H} = \sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}, \quad K = \frac{DD'' - D'^2}{H^2}. \tag{C}$$

Let S be referred to a general system of parametric lines and take for lines of reference upon S_1 the corresponding system; then

$$dx\, dx_1 + dy\, dy_1 + dz\, dz_1 = 0 \tag{1}$$

may be replaced by the three equations:

$$\sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial v} = 0, \quad \sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial v} + \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial u} = 0. \tag{2}$$

We shall make use of Weingarten's* function ϕ, defined by the equations

$$\sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial v} = \phi H, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial u} = -\phi H, \tag{3}$$

and which Bianchi† has called the *characteristic function.* By employing the function thus defined, the differential quotients of x_1 can be expressed in the form‡

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{D\left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v}\right) - D'\left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u}\right)}{HK}, \\ \frac{\partial x_1}{\partial v} &= \frac{D'\left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v}\right) - D''\left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u}\right)}{HK}, \end{aligned} \tag{4}$$

and analogous expressions in y_1 and z_1. From these forms it is seen that as soon as ϕ is known, a surface S_1, which corresponds to S by orthogonality of linear elements, can be found by quadratures; and, moreover, such function ϕ, if it exists, must satisfy the partial differential equation of the second order:

$$\frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{D''\frac{\partial \phi}{\partial u} - D'\frac{\partial \phi}{\partial v}}{HK}\right) + \frac{\partial}{\partial v}\left(\frac{D\frac{\partial \phi}{\partial v} - D'\frac{\partial \phi}{\partial u}}{HK}\right)\right] = \frac{2FD' - ED'' - GD}{H^2}\phi, \tag{5}$$

* Crelle, 100, p. 300. Lezioni, p. 276. ‡ Crelle, 100, p. 303.

which is known as the *characteristic equation.* Bianchi* shows that the characteristic equation can be put in the form

$$\begin{aligned} D''\Big[\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{E}\phi\Big] \\ - 2D'\Big[\frac{\partial^2\phi}{\partial u\,\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\Big] \\ + D\Big[\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{G}\phi\Big] = 0, \quad (5') \end{aligned}$$

where the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}'$ are formed with respect to the square of the linear element of the spherical representation of S, and, furthermore, that to every solution ϕ of this equation there corresponds a solution of the problem.

From the form of this equation we see that the differential equation of its characteristics is

$$Ddu^2 + 2D'du\,dv + D''dv^2 = 0;$$

hence we have the following theorem of Lecornu :†

The asymptotic lines of a surface are the characteristics of its infinitesimal deformation.

In (4) replace the partial differential quotients of X by their expressions in terms of $\frac{\partial x}{\partial u}, \ldots, \frac{\partial z}{\partial v}$; then

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\phi}{H}\Big(F\frac{\partial x}{\partial u} - E\frac{\partial x}{\partial v}\Big) - \frac{X}{HK}\Big(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\Big), \\ \frac{\partial x_1}{\partial v} &= \frac{\phi}{H}\Big(G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}\Big) - \frac{X}{HK}\Big(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\Big), \end{aligned} \quad (6)$$

and similar ones in y_1 and z_1.

Denoting by E_1, F_1, G_1 the first fundamental coefficients of S_1, we have, by definition:

$$E_1 = \sum\Big(\frac{\partial x_1}{\partial u}\Big)^2, \quad F_1 = \sum\frac{\partial x_1}{\partial u}\,\frac{\partial x_1}{\partial v}, \quad G_1 = \sum\Big(\frac{\partial x_1}{\partial v}\Big)^2,$$

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* Bianchi, Lezioni, p. 277.

† "Sur l'équilibre des surfaces flexibles et inextensibles" (Journal de l'Ecole Polytechnique, t. XXIX).

and when $\frac{\partial x_1}{\partial u}$, etc., are replaced by their expressions from (6), we get

$$\left.\begin{aligned} E_1 &= E\phi^2 + \frac{1}{H^2K^2}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)^2, \\ F_1 &= F\phi^2 + \frac{1}{H^2K^2}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right), \\ G_1 &= G\phi^2 + \frac{1}{H^2K^2}\left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right)^2, \end{aligned}\right\} \tag{7}$$

then the square of the linear element of S_1 takes the form

$$ds_1^2 = \phi^2 ds^2 + \frac{1}{H^2K^2}\left[\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)du + \left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right)dv\right]^2, \tag{8}$$

and this can be written

$$ds_1^2 = \phi^2 ds^2 + \lambda^2 d\theta^2,$$

where λ and θ are functions of u and v, whose forms depend upon the surface S and the nature of the deformation, that is, upon the character of ϕ. Comparing this expression with (8), we see that λ must satisfy the condition

$$\frac{\partial}{\partial v}\left[\frac{D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}}{HK\lambda}\right] = \frac{\partial}{\partial u}\left[\frac{D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}}{HK\lambda}\right]. \tag{9}$$

If these expressions are developed and in the reduction use is made of the characteristic equation (5), it is seen that the function λ must satisfy the partial differential equation

$$\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\frac{\partial\lambda}{\partial v} + \left(D''\frac{\partial\phi}{\partial u} - D'\frac{\partial\phi}{\partial v}\right)\frac{\partial\lambda}{\partial u} + K(ED'' + GD - 2FD')\phi\lambda = 0, \tag{10}$$

which is seen to involve the fundamental coefficients of S and the characteristic function ϕ. Having obtained a solution of this equation, the corresponding value of θ can be found by quadratures, since

$$\frac{\partial\theta}{\partial u} = \frac{D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}}{HK\lambda}, \qquad \frac{\partial\theta}{\partial v} = \frac{D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}}{HK\lambda}.$$

From the form of the equation, it follows that (10) is satisfied identically when $\phi = 0$. In this case S_1 reduces to a point.

In the special case where S is a minimal surface, the coefficient of λ in (10) vanishes, and hence this equation is satisfied by

$$\lambda = \text{const.}$$

Since the case where this constant is unity is perfectly general, it is clear that the square of the linear element of S_1 can take the form only when S is a minimal surface,

$$ds_1^2 = \phi^2 ds^2 + d\theta^2,$$

Conversely, when the square of the linear element of S_1 can be put in this form S is a minimal surface.

Write $$A_1 = \frac{\partial(y_1, z_1)}{\partial(u, v)}, \quad B_1 = \frac{\partial(z_1, x_1)}{\partial(u, v)}, \quad C_1 = \frac{\partial(x_1, y_1)}{\partial(u, v)};$$

on introducing, for the sake of brevity, two functions, M and N, defined by

$$M = D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}, \quad N = D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}, \tag{11}$$

the above functions can be expressed, by means of (6), in the form

$$\begin{aligned} A_1 &= A\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial x}{\partial v} - N\frac{\partial x}{\partial u}\right), \\ B_1 &= B\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial y}{\partial v} - N\frac{\partial y}{\partial u}\right), \\ C_1 &= C\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial z}{\partial v} - N\frac{\partial z}{\partial u}\right), \end{aligned} \tag{12}$$

and

$$H_1^2 = A_1^2 + B_1^2 + C_1^2 = H^2\phi^4 + \frac{\phi^2}{H^2K^2}(EN^2 - 2FMN + M^2G). \tag{13}$$

For the sake of brevity, write

$$\delta = \pm\sqrt{H^4K^2\phi^2 + (EN^2 - 2FMN + GM^2)} = \frac{H_1HK}{\phi}, \tag{14}$$

then

$$X_1 = \frac{A_1}{H_1} = \frac{HK\phi A - \left(M\frac{\partial x}{\partial v} - N\frac{\partial x}{\partial u}\right)}{\delta}, \tag{15}$$

and similar expressions for Y_1 and Z_1.

From (6) we get

$$\frac{\partial x_1}{\partial u} = \frac{\phi}{H}\left(F\frac{\partial x}{\partial u} - E\frac{\partial x}{\partial v}\right) - \frac{XM}{HK};$$

differentiating with respect to v,

$$\frac{\partial^2 x_1}{\partial u\,\partial v} = \frac{\partial x}{\partial u}\Big[F\frac{\partial}{\partial v}\frac{\phi}{H} + \frac{\phi}{H}\Big(\frac{\partial F}{\partial v} + F\begin{Bmatrix}12\\1\end{Bmatrix} - E\begin{Bmatrix}22\\1\end{Bmatrix}\Big)\Big] - X\frac{\partial}{\partial v}\frac{M}{\mathcal{H}}$$
$$+ \frac{\partial x}{\partial v}\Big[-E\frac{\partial}{\partial v}\frac{\phi}{H} + \frac{\phi}{H}\Big(-\frac{\partial E}{\partial v} + F\begin{Bmatrix}12\\2\end{Bmatrix} - E\begin{Bmatrix}22\\2\end{Bmatrix}\Big)\Big]$$
$$+ \frac{X\phi}{H}(FD' - ED'') - \frac{M}{\mathcal{H}}\frac{\partial X}{\partial v}. \tag{16}$$

where the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}$ are formed with respect to the square of the linear element of S.

Again, from the first of formulæ (4) we get

$$\frac{\partial^2 x_1}{\partial u\,\partial v} = \Big[\frac{\partial}{\partial v}\Big(\frac{\phi D}{HK}\Big) + \frac{\phi D}{HK}\begin{Bmatrix}22\\2\end{Bmatrix}' - \frac{\phi D'}{HK}\begin{Bmatrix}11\\2\end{Bmatrix}' - \frac{M}{HK}\Big]\frac{\partial X}{\partial v}$$
$$+ \Big[\frac{\phi D}{HK}\begin{Bmatrix}22\\1\end{Bmatrix}' - \frac{\partial}{\partial v}\Big(\frac{D'\phi}{HK}\Big) - \frac{\phi D'}{HK}\begin{Bmatrix}11\\1\end{Bmatrix}'\Big]\frac{\partial X}{\partial u}$$
$$- \Big[\frac{\partial}{\partial v}\Big(\frac{M}{HK}\Big) + \frac{\phi}{HK}(D\mathcal{G} + \mathfrak{E}D')\Big]X, \tag{17}$$

from which it follows that

$$\sum X\frac{\partial^2 x_1}{\partial u\,\partial v} = -\Big[\frac{\partial}{\partial v}\Big(\frac{M}{HK}\Big) + \frac{\phi}{H}(-D'F + ED'')\Big]. \tag{18}$$

Making use of the second of (4), we find

$$\sum X\frac{\partial^2 x_1}{\partial u\,\partial v} = -\Big[\frac{\partial}{\partial u}\Big(\frac{N}{HK}\Big) + \frac{\phi}{H}(-GD + FD')\Big], \tag{19}$$

and in a similar way

$$\left.\begin{aligned}\sum X\frac{\partial^2 x_1}{\partial u^2} &= -\Big[\frac{\partial}{\partial u}\Big(\frac{M}{HK}\Big) + \frac{\phi}{H}(ED' - FD)\Big],\\ \sum X\frac{\partial^2 x_1}{\partial v^2} &= -\Big[\frac{\partial}{\partial v}\Big(\frac{N}{HK}\Big) + \frac{\phi}{H}(FD'' - GD')\Big].\end{aligned}\right\} \tag{20}$$

We are in a position now to calculate the second fundamental coefficients D_1, D_1', D_1'' of S_1. By definition,

$$D_1' = \sum X_1\frac{\partial^2 x_1}{\partial u\,\partial v} = \sum\frac{A_1}{H_1}\frac{\partial^2 x'}{\partial u\,\partial v}.$$

Substituting the expressions for X_1 and $\frac{\partial^2 x'}{\partial u\,\partial v}$ as given by (15) and (16), and

making use of (18), we can reduce this to the form:

$$D_1' = \frac{\phi H}{\delta}\Big[\phi K(FD' - ED'') - HK\frac{\partial}{\partial v}\frac{M}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\Big].$$

By differentiating the second of formulæ (6) with respect to u, another expression for $\frac{\partial^2 x_1}{\partial u\,\partial v}$ can be obtained. Proceeding with this as in the previous case, and making use of (19), we find

$$D_1' = \frac{\phi H}{\delta}\Big[\phi K(GD - FD') - HK\frac{\partial}{\partial u}\frac{N}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\Big].$$

From these two expressions for D_1', those for D_1 and D_1'' can be written by analogy. Replacing δ by its equivalent (14), we have the following expressions for the second fundamental coefficients of S_1:

$$\left.\begin{aligned}
D_1 &= \frac{\phi^2}{H_1K}\Big[\phi K(FD - ED') - HK\frac{\partial}{\partial u}\frac{M}{HK} + M\begin{Bmatrix}11\\1\end{Bmatrix} + N\begin{Bmatrix}11\\2\end{Bmatrix}\Big],\\
D_1' &= \frac{\phi^2}{H_1K}\Big[\phi K(FD' - ED'') - HK\frac{\partial}{\partial v}\frac{M}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\Big],\\
&= \frac{\phi^2}{H_1K}\Big[\phi K(GD - FD') - HK\frac{\partial}{\partial u}\frac{N}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\Big],\\
D_1'' &= \frac{\phi^2}{H_1K}\Big[\phi K(GD' - FD'') - HK\frac{\partial}{\partial v}\frac{N}{HK} + M\begin{Bmatrix}22\\1\end{Bmatrix} + N\begin{Bmatrix}22\\2\end{Bmatrix}\Big].
\end{aligned}\right\} \quad (21)$$

By introducing the values for the partial differential quotients of D, D', D'', as given by the Weingarten formulæ, these expressions can be brought to the form

$$\left.\begin{aligned}
D_1 &= \frac{\phi^2}{H_1K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{E}\phi\Big)\\
&\qquad - D\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\Big)\Big],\\
D_1' &= \frac{\phi^2}{H_1K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\Big)\\
&\qquad - D\Big(\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{G}\phi\Big)\Big],\\
&= \frac{\phi^2}{H_1K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{E}\phi\Big)\\
&\qquad - D'\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\Big)\Big],\\
D_1'' &= \frac{\phi^2}{H_1K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u\,\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\Big)\\
&\qquad - D'\Big(\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathfrak{G}\phi\Big)\Big].
\end{aligned}\right\} \quad (22)$$

These expressions can be put in a simpler and more convenient form by the introduction of the fundamental coefficients of a third surface S_0, which is the *associate* surface of S in an infinitesimal deformation of the latter. As defined by Bianchi,* associate surfaces are such that the normals to the surfaces at corresponding points are parallel and the characteristic function of an infinitesimal deformation of one is equal to the distance from the origin of coordinates to the tangent plane to the other. Denote by D_0, D_0', D_0'' the second fundamental coefficients of S_0; then*

$$\begin{aligned} D_0 &= -\left(\frac{\partial^2\phi}{\partial u^2} - \left\{\begin{matrix}11\\1\end{matrix}\right\}'\frac{\partial\phi}{\partial u} - \left\{\begin{matrix}11\\2\end{matrix}\right\}'\frac{\partial\phi}{\partial v} + \mathfrak{E}\phi\right), \\ D_0' &= -\left(\frac{\partial^2\phi}{\partial u\,\partial v} - \left\{\begin{matrix}12\\1\end{matrix}\right\}'\frac{\partial\phi}{\partial u} - \left\{\begin{matrix}12\\2\end{matrix}\right\}'\frac{\partial\phi}{\partial v} + \mathfrak{F}\phi\right), \\ D_0'' &= -\left(\frac{\partial^2\phi}{\partial v^2} - \left\{\begin{matrix}22\\1\end{matrix}\right\}'\frac{\partial\phi}{\partial u} - \left\{\begin{matrix}22\\2\end{matrix}\right\}'\frac{\partial\phi}{\partial v} + \mathfrak{G}\phi\right). \end{aligned} \tag{23}$$

By means of these expressions, the formulæ (22) can be written:

$$\begin{aligned} D_1 &= \frac{\phi^2}{H_1K}(DD_0' - D_0D'), \\ D_1' &= \frac{\phi^2}{H_1K}(DD_0'' - D_0'D') = \frac{\phi^2}{H_1K}(D'D_0' - D''D_0), \\ D_1'' &= \frac{\phi^2}{H_1K}(D'D_0'' - D_0'D''). \end{aligned} \tag{24}$$

From these values it can be shown at once that

$$DD_1'' + D_1D'' - 2D'D_1' = 0. \tag{25}$$

Hence,

When two surfaces correspond by orthogonality of linear elements, asymptotic lines on one correspond to a conjugate system on the other.†

When two such surfaces are referred to this system of lines, the equations (24) reduce to a form from which it is readily seen that when S is a surface of positive curvature, S_1 is a surface of negative curvature; and when S has negative curvature at a point, the curvature of S_1 at the corresponding point can be positive or negative.

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*Lezioni, p. 278. †Bianchi, p. 284.

Since the expressions for D_1, D_1', D_1'' are linear in the second fundamental coefficients of either S or S_0, the latter can be readily found by solving equations (24). Thus:

$$\left.\begin{aligned} D_0 &= \frac{H_1H^2}{\phi^2}(D_1D' - D_1'D), \\ D_0' &= \frac{H_1H^2}{\phi^2}(D_1D'' - D_1'D') = \frac{H_1H^2}{\phi^2}(D_1'D' - D''D), \\ D_0'' &= \frac{H_1H^2}{\phi^2}(D_1'D'' - D_1''D'). \end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned} D &= \frac{H_1H_0^2K}{\phi^2K_0}(D_1'D_0 - D_1D_0'), \\ D' &= \frac{H_1H_0^2K}{\phi^2K_0}(D_1'D_0' - D_1D_0'') = \frac{H_1H_0^2K}{\phi^2K_0}(D_1''D_0 - D_1'D_0'), \\ D'' &= \frac{H_1H_0^2K}{\phi^2K_0}(D_1''D_0' - D_1'D_0''), \end{aligned}\right\} \tag{27}$$

where H_0 and K are the functions for S_0 corresponding to H and K for S.

From (24) and (26) we get at once

$$\left.\begin{aligned} D_0D_1'' + D_0''D_1 - 2D_1'D_0' &= 0, \\ D_0D'' + D_0''D - 2D_0'D' &= 0. \end{aligned}\right\} \tag{28}$$

Recalling the interpretation of equation (25), we have the theorem:

The asymptotic lines on any one of the three surfaces, S, S_1, S_0, correspond to the system of lines on the other two surfaces which is conjugate for both.

When S is referred to the conjugate system corresponding to the asymptotic lines on S_0, the following relations result, since tangent planes to the two surfaces are parallel:

$$\frac{\partial x}{\partial u} = \frac{D}{D_0'}\frac{\partial x_0}{\partial v}, \quad \frac{\partial x}{\partial v} = \frac{D''}{D_0'}\frac{\partial x_0}{\partial u}, \tag{29}$$

and analogous expressions in y and z. From these follow

$$E = \frac{D^2}{D_0'^2}G_0, \quad F = \frac{DD''}{D_0'^2}F_0, \quad G = \frac{D''^2}{D_0'^2}E_0. \tag{30}$$

From (29) it is manifest that the asymptotic directions on S_0 are parallel to those of the corresponding conjugate systems on S. Again, since S and S_0 correspond by parallelism of tangent planes, the directions of the corresponding conjugate systems on the two surfaces are parallel.

In the special case where the lines of curvature on S correspond to a conjugate system on S_1, it is readily seen from above remarks that the asymptotic lines on S_0 form an orthogonal system. Since the converse is also true, we have the result due to Weingarten:*

The necessary and sufficient condition that the lines of curvature on S correspond to a conjugate system on S_1 is that S_0 be a minimal surface, or, what is the same thing, the spherical representation of the former be isothermal.

An expression for the total curvature of S_1 can be obtained at once by means of (24). It is

$$K_1 = \frac{H^2\phi^4 H_0^2}{H_1^4}\,\frac{K_0}{K}. \tag{31}$$

From this expression and a previous paragraph we get the theorem:

When S is a surface of positive curvature, S_1 and S_0 are of negative curvature; when S is of negative curvature, S_1 and S_0 have opposite kinds of curvature at corresponding points.†

The general expression for the mean curvature of S_1 can be found from formulæ (7) and (24); however, we will consider only the special case where the parametric lines on S_1 correspond to asymptotic lines on S_0. Recalling that in general

$$L_1 = \frac{E_1 D_1'' + G_1 D_1 - 2F_1 D_1'}{H_1^2}, \tag{32}$$

where L_1 denotes the mean curvature of S_1, we find in the case mentioned,

$$L_1 = \frac{\phi^2 D_0'}{H_1^3 K}\left\{\phi^2 (GD - ED'') + \frac{D_0'^2}{\mathcal{K}^2}\left[\frac{DG}{E_0}\left(\frac{\partial\phi}{\partial u}\right)^2 - \frac{ED''}{G_0}\left(\frac{\partial\phi}{\partial v}\right)^2\right]\right\}. \tag{33}$$

The fundamental coefficients of the spherical representation of S_1 can be found directly from formulæ (7) and (24); they are

$$\begin{aligned}
\mathcal{E}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 E_0 + \frac{1}{H_0^2 K_0^2}\left(D_0\frac{\partial\phi}{\partial v} - D_0'\frac{\partial\phi}{\partial u}\right)^2\right],\\
\mathcal{F}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 F_0 + \frac{1}{H_0^2 K_0^2}\left(D_0\frac{\partial\phi}{\partial v} - D_0'\frac{\partial\phi}{\partial u}\right)\left(D_0'\frac{\partial\phi}{\partial v} - D_0''\frac{\partial\phi}{\partial u}\right)\right],\\
\mathcal{G}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 G_0 + \frac{1}{H_0^2 K_0^2}\left(D_0'\frac{\partial\phi}{\partial v} - D_0''\frac{\partial\phi}{\partial u}\right)^2\right].
\end{aligned} \tag{34}$$

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* Sitzungsberichte der König. Akademie zu Berlin, 1886; Bianchi, p. 285.

† The ideas of this theorem are set forth by Bianchi, pp. 279, 286.

We shall denote by $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_1$ the Christoffel symbols formed with respect to the square of the linear element of S_1. Consider first $\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1$. Substituting the values of E_1, F_1, G_1 from (7) and making some straightforward reductions, we get

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1 = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{ND'}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} [D'(MG + NF) D_0' - NFD'' D_0 - MGDD_0''].$$

The value for $\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1$ can be written down at once by analogy. By proceeding in ways similar to the above, the expressions for $\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1$ and $\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1$ can be found, and then $\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1$ and $\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1$ follow by analogy. Having effected these operations, we get the following:

$$\left.\begin{aligned} \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} - \frac{ND}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} (DD_0' - D_0 D')(NF - GM), \\ \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} - \frac{MD}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} (D'D_0 - DD_0')(EN - FM), \end{aligned} \right\} \tag{35}$$

$$\left.\begin{aligned} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{ND'}{(DD'' - D'^2)\phi} \\ &\qquad + \frac{\phi^2}{\mathcal{K}^2 H_1^2} [(MG + NF) D_0' D' - NFD'' D_0 - MGDD_0''], \\ \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} + \frac{MD'}{(DD'' - D'^2)\phi} \\ &\qquad - \frac{\phi^2}{\mathcal{K}^2 H_1^2} [(NE + MF) D_0' D' - NED'' D_0 - MFDD_0''], \end{aligned} \right\} \tag{36}$$

$$\left.\begin{aligned} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} + \frac{ND''}{(DD'' - D'^2)\phi} - \frac{\phi^2}{\mathcal{K}^2 H_1^2} [(FN - GM)(D'' D_0' - D' D_0'')], \\ \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} + \frac{MD''}{(DD'' - D'^2)\phi} - \frac{\phi^2}{\mathcal{K}^2 H_1^2} [(FM - EN)(D'' D_0' - D' D_0'')]. \end{aligned} \right\} \tag{37}$$

Consider the case where S_1 is referred to the conjugate system corresponding to the asymptotic lines on S; then

$$D = D'' = D_0' = D_1' = 0.$$

From (36) we have, on replacing M and N by their values,

$$\left\{\begin{matrix}12\\1\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\1\end{matrix}\right\} + \frac{\partial \log \phi}{\partial v}, \quad \left\{\begin{matrix}12\\2\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\2\end{matrix}\right\} + \frac{\partial \log \phi}{\partial u}.$$

Since S is referred to its asymptotic lines, we must have

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\} = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\}.$$

Hence, if the above equations are differentiated with respect to u and v respectively, and this condition is taken into account, it is necessary that

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\}_1 = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\1\end{matrix}\right\}_1.$$

But when this is true, the equation of the point coordinates of S_1, referred to this conjugate system, namely,

$$\frac{\partial^2\theta}{\partial u\,\partial v} - \left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{\partial\theta}{\partial u} - \left\{\begin{matrix}12\\2\end{matrix}\right\}_1\frac{\partial\theta}{\partial v} = 0,$$

has equal invariants. Since this result is entirely reciprocal as regards S and S_1, we have the following theorem:*

If two surfaces S and S_1 correspond by orthogonality of linear elements and one of the surfaces is referred to its asymptotic lines, the point equation relative to the corresponding conjugate system on the other has equal invariants.

Suppose S and S_1 are referred to the system of lines which is conjugate for both; then

$$D' = D_1' = D_0 = D_0'' = 0.$$

From (36) we have

$$\left\{\begin{matrix}12\\1\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\1\end{matrix}\right\}, \quad \left\{\begin{matrix}12\\2\end{matrix}\right\}_1 = \left\{\begin{matrix}12\\2\end{matrix}\right\},$$

that is, the point coordinates of S_1 satisfy the same equation as those of S, namely,

$$\frac{\partial^2\theta}{\partial u\,\partial v} - \left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{\partial\theta}{\partial u} - \left\{\begin{matrix}12\\2\end{matrix}\right\}\frac{\partial\theta}{\partial v} = 0.$$

Again, we have that the tangential equation of the surface S, referred to this conjugate system, is

$$\frac{\partial^2\phi}{\partial u\,\partial v} - \left\{\begin{matrix}12\\1\end{matrix}\right\}'\frac{\partial\phi}{\partial u} - \left\{\begin{matrix}12\\2\end{matrix}\right\}'\frac{\partial\phi}{\partial v} + \mathscr{H}\phi = 0,$$

* Darboux, Leçons, t. IV, p. 50; Bianchi, p. 286.

but since the spherical representation of S is also that of S_0 and the latter is referred to asymptotic lines, it follows that the tangential equation has equal invariants. Inasmuch as the relation between S and S_1 is reciprocal, this result is also true for S_1. Hence:

*If two surfaces correspond by orthogonality of linear elements they have the same point equation when the parametric lines on each constitute the conjugate system corresponding on the two surfaces; and the tangential equations of both surfaces relatively to this system have equal invariants.**

Darboux† has shown that when the condition

$$\sum dx\, dx_1 = 0$$

is satisfied by the cartesian coordinates of corresponding points on S and S_1 there exist three quantities x_0, y_0, z_0 which are such that

$$\begin{aligned} dx_1 &= z_0 dy - y_0 dz, \\ dy_1 &= x_0 dz - z_0 dx, \\ dz_1 &= y_0 dx - x_0 dy, \end{aligned} \tag{38}$$

and Genty‡ has proved that the surface which has x_0, y_0, z_0 for its point coordinates is S_0, the associate surface of S in an infinitesimal deformation of the latter.

Multiplying these equations by x_0, y_0, z_0 respectively and adding, we have

$$x_0 dx_1 + y_0 dy_1 + z_0 dz_1 = 0,$$

from which it follows that

$$\sum x_0 \frac{\partial x_1}{\partial u} = 0, \quad \sum x_0 \frac{\partial x_1}{\partial v} = 0.$$

But

$$\sum X_1 \frac{\partial x_1}{\partial u} = 0, \quad \sum X_1 \frac{\partial x}{\partial v} = 0,$$

hence,

$$\frac{x_0}{X_1} = \frac{y_0}{Y_1} = \frac{z_0}{Z_1} = r_0,$$

where r_0 is the radius vector of S_0 at the point (x_0, y_0, z_0). From this it follows

* The second part of this theorem is given by Darboux, IV, p. 71.

† Leçons, t. IV, p. 8.

‡ Toulouse Annales, t. IX, E.

that *the radius vector of S_0 at the point M_0 is parallel to the normal to S_1 at the corresponding point.**

And as a consequence of this:

The necessary and sufficient condition that the tangent planes at corresponding points of S and S_1 be parallel is that S_0 be a sphere.

III.

Consider the surface S', for which the cartesian coordinates of a point are defined by

$$x' = x + \tau x_1,\ y' = y + \tau y_1,\ z' = z + \tau z_1, \tag{1}$$

where (x, y, z) and (x_1, y_1, z_1) are the coordinates of corresponding points on S and S_1 and τ is a constant which, for the present, may have any finite value.

If ds' denote an element of length on S', the square of this linear element may be written in the form

$$ds'^2 = E'\,du^2 + 2F'\,dudv + G'\,dv_1^2 \tag{2}$$

where, in consequence of (1),

$$E' = E + \tau^2 E_1,\ F' = F + \tau^2 F_1,\ G' = G + \tau^2 G_1, \tag{3}$$

or by (II, 7)

$$\left.\begin{aligned} E' &= E(1 + \tau^2\phi^2) + \frac{\tau^2 M^2}{H^2K^2}, \\ F' &= F(1 + \tau^2\phi^2) + \frac{\tau^2 MN}{H^2K^2}, \\ G' &= G(1 + \tau^2\phi^2) + \frac{\tau^2 N^2}{H^2K^2}. \end{aligned}\right\} \tag{3'}$$

From (3) and (3′) we obtain directly

$$H'^2 = (H^2 + \frac{\tau^2}{\phi^2} H_1^2)(1 + \tau^2\phi^2), \tag{4}$$

and

$$H'^2 = (1 + \tau^2\phi^2)\Big[H^2(1 + \tau^2\phi^2) + \frac{\tau^2(EN^2 - 2FMN + GM^2)}{H^2K^2}\Big]. \tag{5}$$

Put

$$A' = \frac{\partial(y', z')}{\partial(u, v)},\quad B' = \frac{\partial(z', x')}{\partial(u, v)},\quad C' = \frac{\partial(x', y')}{\partial(u, v)}; \tag{6}$$

* This theorem is essentially the same as that given by Darboux, IV, p. 59.

then if the differential quotients of x', y', z' be replaced by their expressions from (1) and formulæ (II, 6) be used, A' can be expressed in the form

$$A' = A + \frac{\tau}{K}\,\frac{\partial(\phi, X)}{\partial(u, v)} + \tau^2 A_1,$$

and the expressions for B' and C' are similar.

If the second fundamental coefficients of S' be denoted by $D^{(1)}$, $D'^{(1)}$, $D''^{(1)}$, the first is given by

$$D^{(1)} = \sum X' \frac{\partial^2 x'}{\partial u^2} = \sum \frac{A'}{H'}\left(\frac{\partial^2 x}{\partial u^2} + \tau \frac{\partial^2 x_1}{\partial u^2}\right).$$

Making use of formulæ (II, 13 and 20) we can bring this to the form

$$D^{(1)} = \frac{1}{H'}\left[\frac{D}{H}\left(H^2 + \frac{H_1^2}{\phi^2}\tau^2\right) + H_1 D_1 \tau\left(\frac{1}{\phi^2} + \tau^2\right)\right]. \tag{7}$$

The expressions for $D'^{(1)}$ and $D''^{(1)}$ follow by analogy; they are

$$\begin{aligned} D'^{(1)} &= \frac{1}{H'}\left[\frac{D'}{H}\left(H^2 + \frac{H_1^2}{\phi^2}\tau^2\right) + H_1 D_1' \tau\left(\frac{1}{\phi^2} + \tau^2\right)\right], \\ D''^{(1)} &= \frac{1}{H'}\left[\frac{D''}{H}\left(H^2 + \frac{H_1^2}{\phi^2}\tau^2\right) + H_1 D_1'' \tau\left(\frac{1}{\phi^2} + \tau^2\right)\right]. \end{aligned} \tag{7}$$

From these expressions we can readily find an expression for the total curvature of S' by taking account of the condition (II, 25). Thus:

$$K' = \frac{K}{(1+\tau^2\phi^2)^2} + \frac{\tau^2 H_1^4 K_1\left(\tau^2 + \frac{1}{\phi^2}\right)^2}{(1+\tau^2\phi^2)^2\left(H^2 + H_1^2\frac{\tau^2}{\phi^2}\right)}. \tag{8}$$

In the special case where S_1 is a sphere, the distance between corresponding points on the two applicable surfaces Σ_1 and Σ_2, defined by

$$\begin{aligned} \xi_1 &= x + \tau x_1, \quad \eta_1 = y + \tau y_1, \quad \zeta_1 = z + \tau z_1, \\ \xi_2 &= x - \tau x_1, \quad \eta_2 = y - \tau y_1, \quad \zeta_2 = z - \tau z_1, \end{aligned}$$

where $\tau = \sqrt{x_1^2 + y_1^2 + z_1^2}$, is constant.

It is evident that in this case the surface Σ_1 is the same as S' in the beginning of this section, and that the coefficients of Σ_2 are gotten by changing τ into

$-\tau$ in those formulæ. From the expressions for E', F', G', it is seen that the only case in which Σ_1 and Σ_2 could be applicable to S would arise from the value

$$\phi = 0,$$

which simply means that Σ_1, Σ_2 and S coincide.

In order that Σ_1 and Σ_2 be the same surface to a displacement *près*, it is necessary and sufficient that their second fundamental coefficients are equal. Referring to formulæ (7), we remark that we must have

$$\frac{1}{\phi^2} + \tau^2 = 0,$$

that is, $\phi = \text{const.}$, from which it follows that S_0 may be either a sphere or a plane. However, the former case must be excluded, since S_0 and S_1 cannot be spheres simultaneously. But when S_0 is a plane, S also is a plane; hence,

The necessary and sufficient condition that the two surfaces Σ_1 and Σ_2 be the same surface to a displacement près, is that S be a plane.

The formulæ which have been deduced in this section are evidently independent of the nature of the constant quantity τ. If, then, τ is replaced by ε, which is a small constant whose powers higher than the first are negligible, the surface S' will be infinitely near S and will be obtained by deforming points of the latter along lines whose direction cosines are proportional to the cartesian coordinates of the corresponding points on S_1. With Darboux we shall call the lines, in the direction of which the deformation is effected, the *directrices of the deformation.*

The square of the linear element of S' is

$$ds'^2 = dx^2 + dy^2 + dz^2 + \varepsilon^2 (dx_1^2 + dy_1^2 + dz_1^2),$$

that is, S' is applicable to S to terms of the second order *près*, which, by hypothesis, are neglected. Hence,

$$E' = E, \quad F' = F, \quad G' = G. \tag{9}$$

In this case the expressions for the second fundamental coefficients of S' become

$$D^{(1)} = D + \varepsilon \frac{H_1 D_1}{\phi^2 H}, \quad D'^{(1)} = D' + \varepsilon \frac{H_1 D_1'}{\phi^2 H}, \quad D''^{(1)} = D'' + \varepsilon \frac{H_1 D_1''}{\phi^2 H}, \tag{10}$$

and we have for the total and mean curvature of S' respectively,

$$K' = K, \quad L' = L + \varepsilon \frac{H_1 (GD_1 + ED_1'' - 2FD_1')}{\phi^2 H^3},$$

the former being simply an expression of the Gauss theorem.

The expressions for the fundamental coefficients of the spherical representation of S', denoted by $\mathfrak{E}'$, $\mathfrak{F}'$, $\mathfrak{G}'$, are readily calculated; they are

$$\mathfrak{E}' = \mathfrak{E} + \frac{2H_1\varepsilon}{\phi^2 H^3}[GDD_1 - F(DD_1' + D'D_1) + ED'D_1'],$$

$$\mathfrak{F}' = \mathfrak{F} + \frac{2H_1\varepsilon}{\phi^2 H^3}[G(DD_1' + D_1D') - F(DD_1'' + D''D_1 + 2D'D_1') + E(D'D_1'' + D''D_1')],$$

$$\mathfrak{G}' = \mathfrak{G} + \frac{2H_1\varepsilon}{\phi^2 H^3}[GD'D_1' - F(D'D_1'' + D_1'D'') + ED_1'D''].$$

Replace D_1, D_1', D_1'' by their values as given in (24); then

$$\left.\begin{aligned} \mathfrak{E}' &= \mathfrak{E} + \frac{2\varepsilon}{HK}\left(\mathfrak{E}D_0' - \mathfrak{F}D_0\right), \\ \mathfrak{F}' &= \mathfrak{F} + \frac{2\varepsilon}{HK}\left(\mathfrak{E}D_0'' - \mathfrak{G}D_0\right), \\ \mathfrak{G}' &= \mathfrak{G} + \frac{2\varepsilon}{HK}\left(\mathfrak{F}D_0'' - \mathfrak{G}D_0'\right). \end{aligned}\right\} \tag{11}$$

It is evident from these expressions that the necessary and sufficient condition that the tangent planes at corresponding points of S and S' be parallel is expressed by

$$\frac{\mathfrak{E}}{D_0} = \frac{\mathfrak{F}}{D_0'} = \frac{\mathfrak{G}}{D_0''},$$

or, since the spherical representation of S and S_0 is the same,

$$\frac{\mathfrak{E}_0}{D_0} = \frac{\mathfrak{F}_0}{D_0'} = \frac{\mathfrak{G}_0}{D_0''}.$$

But this condition is satisfied only when S_0 is a plane or a sphere. In the former case, S and S' are planes, that is, S has been subjected to a displacement and not an infinitesimal deformation; hence, this case is excluded. We have then the following theorem:

The necessary and sufficient condition that S and S' correspond by parallelism of tangent planes is that S_0 be a sphere, or what is the same thing, that S and S_1 have such correspondence.

From the expression for $D'^{(1)}$, it is manifest that the conjugate system on S, which corresponds to a conjugate system on S', is that which corresponds to a conjugate system on S_1, or, what is the same thing, to the asymptotic lines on S_0. Hence :*

The conjugate system of lines on S which remains conjugate for a given infinitesimal deformation of the surface corresponds to the asymptotic lines on its associate.

If $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}^{(1)}$ denotes the Christoffel symbols formed with respect to the square of the linear element of S', we have in consequence of (9),

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}^{(1)} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}, \quad \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}^{(1)} = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}.$$

Hence, *when S, S_1 and S' are referred to the double system of lines which are conjugate for all three surfaces, their point coordinates are particular solutions of the same partial differential equation of the second order.*

When the lines of curvature constitute the system which has for its correspondent a conjugate system on S', the latter is also made up of lines of curvature. Recalling a result obtained in a previous section, we have the well-known theorem :†

The necessary and sufficient condition that the lines of curvature of a surface be preserved in an infinitesimal deformation is that the spherical representation of the lines of curvature form an isothermal system.

Guichard has shown‡ that any surface can be brought into correspondence with a plane by orthogonality of linear elements. Hence an infinitesimal deformation of the plane can be effected as soon as any surface whatever has been brought into such correspondence by the means suggested by Guichard. It is evident that S_0 also will be a plane. From formulæ (10), it follows that on the resulting surface the asymptotic lines will correspond to the asymptotic lines on S_1.

If, now, the deformation of any surface is considered for which S_1 is a plane, the formulæ (10) show that the second fundamental coefficients of the original surface and the surface resulting from the deformation are equal, since $\phi \neq 0$ for this case. Hence, when S_1 is a plane, there is no deformation, but only an infinitesimal displacement of S. We shall show that this sufficient condition is also necessary.

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* Bianchi, p. 284. † Ib., p. 285. ‡ Comptes Rendus, t. CXIV, p. 729.

In order that S' differs from S only by an infinitely small displacement, it is necessary that

$$x_1 = c_1 + \gamma y - \beta z, \quad y_1 = c_2 + \alpha z - \gamma x, \quad z_1 = c_3 + \beta x - \alpha y, \tag{12}$$

where $\alpha, \beta, \gamma, c_1, c_2, c_3$ are constants. Eliminating x, y, z from these expressions, we find that S_1 is a plane. Hence,*

The necessary and sufficient condition that S' be S to an infinitely small displacement is that S_1 be a plane.

Differentiating the expressions (12) and comparing them with the formulæ (II, 38), we find

$$x_0 = \alpha, \quad y_0 = \beta, \quad z_0 = \gamma,$$

that is, *when S_1 is a plane, S_0 is a point, and conversely.*

Let S be referred to its asymptotic lines. Since then $D_1' = 0$, in order that the corresponding lines on S' be the asymptotic lines, either

$$D_1 = D_1' = D_1'' = 0,$$

or

$$\varepsilon = 0.$$

It has been shown that in the former case there is only a displacement; in latter case, there has been no deformation. Hence,

The asymptotic lines of a surface are not preserved in any infinitesimal deformation of the surface.

IV.

From the form of the characteristic equation, it is readily seen that only in the case where S is a minimal surface can the characteristic function ϕ be a constant. When ϕ is a constant, S_0 is either a plane or a sphere. In the former case, S is a plane, and it will be remarked that this case is in perfect accord with the results which follow when S_0 is a sphere.

Let S_0 be a sphere of radius unity, that is, $\phi = 1$. Formulæ (II, 7) give

$$E_1 = E, \quad F_1 = F, \quad G_1 = G,$$

hence,

$$ds_1^2 = ds^2,$$

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* Darboux, IV, p. 9.

that is, S and S_1 are applicable surfaces. Since S and S_1 correspond by orthogonality of linear elements, S_1 is the *adjoint minimal* surface of S. This is immediate from the following theorem, as stated by Darboux:*

If two surfaces are applicable to one another by orthogonality of corresponding linear elements, they can only be two minimal surfaces, of which one is the adjoint of the other.

This result is also obtained by remarking that in this case x_0, y_0, z_0 may be replaced by X, Y, Z respectively, so that formulæ (II, 38) become

$$\begin{aligned} dx_1 &= Ydz - Zdy, \\ dy_1 &= Zdx - Xdz, \\ dz_1 &= Xdy - Ydx, \end{aligned}$$

which are the formulæ of Schwarz,† giving the relations between the differentials of the cartesian coordinates of a minimal surface and of its adjoint and the direction cosines of tangent planes.

Since the adjoint of a minimal surface is unique, we have the following theorem as a consequence of the above theorem of Darboux:‡

The necessary and sufficient condition that a surface S be applicable to a surface S_1 which figures in an infinitesimal deformation of the former, is that S_1 correspond to the value unity of the characteristic function; moreover, this is only possible when S is a minimal surface.

From the formulæ (II, 21), we obtain the following relations between the fundamental coefficients of a minimal surface and its adjoint when referred to a general system of curvilinear coordinates.

$$\left.\begin{aligned} D_1 &= \frac{FD - ED'}{H}, \quad D_1'' = \frac{GD' - FD''}{H}, \\ D_1' &= \frac{FD' - ED''}{H} = \frac{GD - FD'}{H}. \end{aligned}\right\} \tag{1}$$

Since S_0 is a sphere,

$$D_0,\ D_0',\ D_0'' = -E_0,\ -F_0,\ -G_0;$$

and remarking that in the present case S_0 is the spherical representation of S, we have from (II, 24) the following relations between the second fundamental

* Leçons, t. I, p. 331.
† "Miscellen aus dem Gebiete der Minimalflachen," Crelle, 80, p. 287.
‡ Bianchi, p. 348.

coefficients of a minimal surface and its adjoint and the coefficients of their common spherical representation:

$$\left.\begin{aligned} D_1 &= \frac{D'\mathcal{E}-D\mathcal{F}}{\mathcal{H}}, \quad D_1'' = \frac{\mathcal{F}D''-\mathcal{G}D'}{\mathcal{H}}, \\ D_1' &= \frac{\mathcal{E}D''-\mathcal{F}D'}{\mathcal{H}} = \frac{\mathcal{F}D'-\mathcal{G}D}{\mathcal{H}}. \end{aligned}\right\} \tag{2}$$

Comparing the two sets of expressions (1) and (2), we find that in the case of a minimal surface,

$$\frac{\mathcal{E}}{E} = \frac{\mathcal{F}}{F} = \frac{\mathcal{G}}{G} = -K; \tag{3}$$

from this it follows that

$$d\sigma^2 = -Kds^2 = -Kds_1^2. \tag{4}$$

In the present case the asymptotic directions on S_0 are given by

$$\mathcal{E}du^2 + 2\mathcal{F}du\,dv + \mathcal{G}dv^2 = 0,$$

that is, lines of length zero. Since the corresponding systems of lines on S and S_1 are conjugate for both surfaces, it follows from (4) that—

*The conjugate system of lines on a minimal surface which corresponds to a conjugate system on its adjoint, is made up of lines of length zero on both.**

When S_0, a sphere, is referred to its lines of length zero, the point coordinates are the following functions of the parameters of these lines:†

$$x_0 = \frac{u+v}{1+uv}, \quad y_0 = i\,\frac{u-v}{1+uv}, \quad z_0 = \frac{uv-1}{1+uv}. \tag{5}$$

However, since S_0 is the spherical representation of S, the direction cosines of the tangent plane to the latter are given by

$$X = \frac{u+v}{1+uv}, \quad Y = i\,\frac{u-v}{1+uv}, \quad Z = \frac{uv-1}{1+uv},$$

where u and v refer to conjugate system on S composed of lines of length zero.

In section III we showed that the necessary and sufficient condition that the tangent planes at corresponding points of S and S' be parallel is that S_0 be a sphere. Since the deformation of S into S' is such that S and S' are applicable,‡ we have that in the case where S_1 *is the adjoint of* S, *the surface* S' *is an associate minimal surface of* S. Conversely, when S and S' are associate minimal sur-

* Darboux, Leçons, t. IV, p. 96. † Ib., I, p. 37. ‡ Darboux, Leçons, t. I, p. 326.

faces, S_0 is a sphere; but we cannot take a special value for its radius: hence it will be necessary to ascertain the relations between S and S_1. From (II, 21) it follows that S_1 also is a minimal surface, and since S_0 is a sphere, the tangent planes to S_1 and S_0 at corresponding points are parallel. Referring to formulæ (II, 7) we get

$$E_1 = E\phi^2, \quad F_1 = F\phi^2, \quad G_1 = G\phi^2.$$

Denote by subscript a the quantities belonging to the adjoint of S, then

$$X_1 = X = X_a, \quad Y_1 = Y = Y_a, \quad Z_1 = Z = Z_a,$$

and

$$E_1 = \phi^2 E = \phi^2 E_a, \quad F_1 = \phi^2 F = \phi^2 F_a, \quad G_1 = \phi^2 G = \phi^2 G_a;$$

and by making use of (II, 24) and (2), we find

$$D_1 = \phi D_a, \quad D_1' = \phi D_a', \quad D_1'' = \phi D_a''.$$

From the last and first of these three systems of equalities, we have, for $\phi = \text{const.}$,

$$\sum \frac{\partial X}{\partial u}\,\frac{\partial x_1}{\partial u} = \phi \sum \frac{\partial X}{\partial u}\,\frac{\partial x_a}{\partial u} = \sum \frac{\partial X}{\partial u}\,\frac{\partial \phi x_a}{\partial u},$$

whence

$$\sum \frac{\partial X}{\partial u}\,\frac{\partial (x_1 - \phi x_a)}{\partial u} = 0, \quad \sum \frac{\partial X}{\partial v}\,\frac{\partial (x_1 - \phi x_a)}{\partial u} = 0.$$

Solving these two equations, we get

$$\frac{\dfrac{\partial (x_1 - \phi x_a)}{\partial u}}{\dfrac{\partial (Y, Z)}{\partial (u, v)}} = \frac{\dfrac{\partial (y_1 - \phi y_a)}{\partial u}}{\dfrac{\partial (Z, X)}{\partial (u, v)}} = \frac{\dfrac{\partial (z_1 - \phi z_a)}{\partial u}}{\dfrac{\partial (X, Y)}{\partial (u, v)}};$$

and hence,

$$\frac{\dfrac{\partial (x_1 - \phi x_a)}{\partial u}}{X} = \frac{\dfrac{\partial (y_1 - \phi y_a)}{\partial u}}{Y} = \frac{\dfrac{\partial (z_1 - \phi z_a)}{\partial u}}{Z}.$$

Since

$$\sum X \frac{\partial x_1}{\partial u} = 0 \quad \text{and} \quad \sum X \frac{\partial x_a \phi}{\partial u} = 0,$$

we have

$$\sum X \frac{\partial (x_1 - x_a \phi)}{\partial u} = 0.$$

Combining this with the above, we get

$$\frac{\partial (x_1 - \phi x_a)}{\partial u} = \frac{\partial (y_1 - \phi y_a)}{\partial u} = \frac{\partial (z_1 - \phi z_a)}{\partial u} = 0,$$

and, in like manner, by considering $\sum \frac{\partial X}{\partial v} \frac{\partial x_1}{\partial v}$,

$$\frac{\partial (x_1 - \phi x_a)}{\partial v} = \frac{\partial (y_1 - \phi y_a)}{\partial v} = \frac{\partial (z_1 - \phi z_a)}{\partial v} = 0.$$

Hence, $$x_1 = \phi x_a + a, \quad y_1 = \phi y_a + b, \quad z_1 = \phi z_a + c,$$

where a, b, c are constants. From this it follows that—

If two surfaces have their tangent planes at corresponding points parallel and their second fundamental coefficients are in constant ratio, the one is a homothetic of the other to within a translation.

Since the infinitesimal deformation of a surface S, where the direction cosines of the directrices of deformation are proportional to the coordinates of a surface homothetic to S_1, is the same as when the direction cosines are proportional to the point coordinates of the latter, we have the following theorem:*

The necessary and sufficient condition that a minimal surface be deformed into a minimal surface associate to itself, is that the direction cosines of the directrices of the infinitesimal deformation be proportional to the cartesian coordinates of the corresponding points on the adjoint minimal surface of the given one.

From Darboux† we have, that if the point coordinates of a minimal surface, referred to its conjugate system of lines of length zero, are given by

$$\begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned}$$

then the associate surfaces of this surface are given by

$$\begin{aligned} x &= e^{i\alpha} A(u) + e^{-i\alpha} A_1(v), \\ y &= e^{i\alpha} B(u) + e^{-i\alpha} B_1(v), \\ z &= e^{i\alpha} C(u) + e^{-i\alpha} C_1(v), \end{aligned}$$

each surface corresponding to a value of α. Furthermore, for the values $\alpha = 0$ and $\alpha = \pi/2$, these give the coordinates of the given surface and its adjoint respectively. If, then, α is allowed to vary continuously, we have a succession of minimal surfaces, applicable to the given one and to each other, extending, as it were, from the given surface to its adjoint. Moreover, from the nature of this process, it follows that this series of surfaces is unique.

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* Bianchi, p. 346.

† Leçons, t. I, p. 322.

We will consider now this series of associate minimal surfaces from the standpoint of infinitesimal deformation. From what precedes, it is clear that the associate surface of this series, infinitely near the given surface and corresponding to a value of α very near zero, can be gotten by deforming S infinitely little along directrices whose direction cosines are proportional to the cartesian coordinates of the adjoint; and, furthermore, it has been shown that this is the only way to get this surface by deformation. Consider in turn this surface and the third one in the series. The latter is an associate of the former and infinitely near it, and hence can be gotten by effecting upon the second surface an infinitesimal deformation, whose directrices have for direction cosines quantities proportional to the cartesian coordinates of the corresponding point on the adjoint of the second surface. Proceeding in this way step by step, the above series of surfaces can be obtained. We have, then, the following result:

The series of minimal surfaces, associate to a given minimal surface, which is obtained by varying continuously a parameter which enters in the expressions of the cartesian coordinates of a point on any surface, can be gotten by deforming the given surface and its successive associates infinitely little in directrices whose cosines are proportional to the point coordinates of the adjoint minimal surface of the surface deformed.

By means of (III, 13), the following relations can be found between the fundamental coefficients of a minimal surface, its adjoint and the associate surface which is the result of a given infinitesimal deformation of this surface:

$$D^{(1)} = D + \frac{D_1}{K}\varepsilon, \quad D'^{(1)} = D' + \frac{D_1'}{K}\varepsilon, \quad D''^{(1)} = D'' + \frac{D_1''}{K}\varepsilon. \tag{6}$$

Consider the case where S is a sphere, and let it be referred to the conjugate system corresponding to asymptotic lines on S_0. Since every conjugate system of lines on a sphere is an orthogonal system, and since the asymptotic directions on S_0 and the corresponding conjugate directrices on S are parallel, it follows that the asymptotic directrices on S_0 must be orthogonal, that is, S_0 is a minimal surface. We have, then, that *the associate surface of a sphere is a minimal surface.**

Recalling that for the sphere

$$D, D', D'' = -E, -F, -G,$$

* Darboux, IV, p. 96.

and substituting these values in the expression for the mean curvature of the surface corresponding to the sphere by orthogonality of linear elements (II, 33), we have

$$L_1 = \frac{\phi^2 D_0''^2}{H_1^3 K} EG\left[\frac{1}{G_0}\left(\frac{\partial\phi}{\partial v}\right)^2 - \frac{1}{E_0}\left(\frac{\partial\phi}{\partial u}\right)^2\right].$$

Since the asymptotic lines form an isothermal system on a minimal surface, we have

$$E_0 = G_0;$$

hence, the necessary and sufficient condition that S_1 be a minimal surface is

$$\frac{\partial\phi}{\partial v} = \pm\frac{\partial\phi}{\partial u},$$

It will be seen that the following discussion will be just as general if we consider only the case where the sign in the above equation is plus. The general integral of this equation is a function of $u+v$; write it $\phi(u+v)$. From the definition of associate surface this function must be of such a form that it is the distance from the origin to the tangent plane to the minimal surface S_0. Since S_0 is referred to its asymptotic lines

$$\mathscr{F}_0 = 0,\ \mathscr{E}_0 = \mathscr{G}_0 = \frac{1}{\rho_0},$$

where

$$\frac{1}{\rho_0^2} = -K_0;$$

hence ϕ must satisfy the equation *

$$\phi'' + \frac{1}{\rho_0}\phi = 0,$$

where ϕ'' denotes the second derivative of ϕ with respect to $u+v$. From the form of this equation it is evident that for such a function ϕ to exist there must be a minimal surface for which ρ_0 is a function of $u+v$. We proceed to the determination of such a surface. Replace F by zero, E and G by ρ_0 and K by $-\frac{1}{\rho_0^2}$ in the Gauss equation,† then it is found that ρ_0 must satisfy the equation

$$\frac{d^2\log\theta}{d(u+v)^2} = \frac{1}{\theta}.$$

This equation is readily integrated and gives

$$\rho_0 = \frac{\alpha^2}{\beta^2}\left(\frac{\beta^2}{2}e^{\frac{u+v}{2\alpha}} + e^{-\frac{u+v}{2\alpha}}\right)^2$$

*Bianchi, p. 137 (87).

†Ib., p. 67 (19).

where α and β are constants. Since the Gauss and Codazzi equations are satisfied in the preceding results, there is a minimal surface which when referred to its asymptotic lines has for the square of its linear element the expression

$$ds^2 = \frac{\alpha^2}{\beta^2}\left(\frac{\beta^2}{2}e^{\frac{u+v}{2\alpha}} + e^{-\frac{u+v}{2\alpha}}\right)^2 (du^2 + dv^2).$$

Recalling the preceding results we have that only when S_0 is this particular minimal surface and S is the sphere of radius unity is S_1 a minimal surface.

Consider again the case where S is a minimal surface. Since the spherical representation of its lines of curvature form an isothermal system, S can be deformed so as to preserve these lines, and, as we have seen, S_0 is a minimal surface in this case. It is natural to inquire whether there is any special relation between the two minimal surfaces S and S_0. Refer the former to its lines of curvature; then

$$ds^2 = \rho\,(du^2 + dv^2),$$

where

$$\rho = \rho_1 = -\rho_2,$$

hence,

$$E = G = \frac{E}{D} = -\frac{G}{D''},$$

from which it follows that

$$D = 1, \quad D'' = -1.$$

The surface S_0 will be referred to its asymptotic lines, hence

$$ds_0^2 = \rho_0\,(du^2 + dv^2);$$

from this we get

$$E_0 = G_0, \quad D_0' = -1,$$

Substituting these values of D, D_0', D'' in formulæ (II, 30), we get

$$E = G_0 = E_0, \quad G = E_0 = G_0.$$

Thus S and S_0 are applicable and we know that asymptotic lines on S_0 correspond to lines of curvature on S. Hence:

The minimal surface which is the associate of a given minimal surface in the infinitesimal deformation of the latter, leaving the lines of curvature unaltered, is the adjoint of the given one.

We shall now discuss the question as to whether it is possible to deform S and S_1 in such a way that the surfaces resulting from the deformation shall be

in correspondence by orthogonality of linear elements. Denote by (x', y', z') and (x_1, y_1', z_1') corresponding points on these new surfaces; then we can write

$$x' = x + \varepsilon x_1, \quad x_1' = x_1 + \varepsilon_1 x_2, \tag{7}$$

and similar expressions for the y's and z's. For these two surfaces to have the desired correspondence, we must have

$$\sum dx'\, dx_1' = \sum dx\, dx_1 + \varepsilon \sum dx_1^2 + \varepsilon_1 \sum dx\, dx_2 + \varepsilon\varepsilon_1 \sum dx_1\, dx_2 = 0.$$

Since

$$\sum dx\, dx_1 = 0 \quad \text{and} \quad \sum dx_1\, dx_2 = 0,$$

it is necessary that

$$\varepsilon \sum dx_1^2 + \varepsilon_1 \sum dx\, dx_2 = 0;$$

that is, the following relations also must hold:

$$\varkappa \sum \left(\frac{\partial x_1}{\partial u}\right)^2 = \sum \frac{\partial x}{\partial u}\,\frac{\partial x_2}{\partial u}, \quad \varkappa \sum \left(\frac{\partial x_1}{\partial v}\right)^2 = \sum \frac{\partial x}{\partial v}\,\frac{\partial x_2}{\partial v},$$

$$2\varkappa \sum \frac{\partial x_1}{\partial u}\,\frac{\partial x_1}{\partial v} = \sum \frac{\partial x}{\partial u}\,\frac{\partial x_2}{\partial v} + \sum \frac{\partial x}{\partial v}\,\frac{\partial x_2}{\partial u}, \tag{8}$$

where $\varkappa = -\dfrac{\varepsilon}{\varepsilon_1}$, a constant.

From a proposition of Cauchy, that the integration by quadratures of a system of linear equations with second members, follows as soon as a general solution of the same system with second members zero has been obtained, it follows that when S_1 has been found, that is, when the system of equations

$$\sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial v} = 0, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial u} + \sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial v} = 0$$

has been integrated, we can integrate by quadratures the systems (8), in which x_1, y_1, z_1 have been given the values obtained by the previous integration. If the integrals so found satisfy, in conjunction with those of the previous system, the condition

$$\sum dx_1\, dx_2 = 0,$$

we know that when the deformation indicated by (7) is effected, the resulting surfaces correspond by orthogonality of linear elements.

That there are such cases can be shown by looking at the question with the means discussed in the preceding pages. For, if we apply the formulæ of section II to the surface S_1, at one time in conjunction with S and the other with S_2, and we denote by ϕ_1 and ϕ_2 the characteristic functions in these two cases, we have (II, 6),

$$\frac{\partial x}{\partial u} = \frac{\phi_1}{H_1}\left(F_1\frac{\partial x_1}{\partial u} - E_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}M_1,$$
$$\frac{\partial x}{\partial v} = \frac{\phi_1}{H_1}\left(G_1\frac{\partial x_1}{\partial u} - F_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}N_1,$$

with similar expressions in y and z; and also,

$$\frac{\partial x_2}{\partial u} = \frac{\phi_2}{H_1}\left(F_1\frac{\partial x_1}{\partial u} - E_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}M_2,$$
$$\frac{\partial x_2}{\partial v} = \frac{\phi_2}{H_1}\left(G_1\frac{\partial x_1}{\partial u} - F_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}N_2.$$

Substituting these values in the above equations, we can reduce them to the following:

$$M_1M_2 + E_1H_1^2K_1^2(\phi_1\phi_2 - \varkappa) = 0,$$
$$N_1N_2 + G_1H_1^2K_1^2(\phi_1\phi_2 - \varkappa) = 0,$$
$$M_1N_2 + M_2N_1 + 2F_1H_1^2K_1^2(\phi_1\phi_2 - \varkappa) = 0,$$

whence

$$\frac{M_1M_2}{E_1} = \frac{N_1N_2}{G_1} = \frac{M_1N_2 + M_2N_1}{2F_1}, \tag{9}$$

or, on writing the expressions for M_1, N_1, M_2, N_2,

$$\frac{\left(D_1\frac{\partial\phi_1}{\partial v} - D_1'\frac{\partial\phi_1}{\partial u}\right)\left(D_1\frac{\partial\phi_2}{\partial v} - D_1'\frac{\partial\phi_2}{\partial u}\right)}{E_1} = \frac{\left(D_1'\frac{\partial\phi_1}{\partial v} - D_1''\frac{\partial\phi_1}{\partial u}\right)\left(D_1'\frac{\partial\phi_2}{\partial v} - D_1''\frac{\partial\phi_2}{\partial u}\right)}{G_1},$$
$$= \frac{\left(D_1\frac{\partial\phi_1}{\partial v} - D_1'\frac{\partial\phi_1}{\partial u}\right)\left(D_1'\frac{\partial\phi_2}{\partial v} - D_1''\frac{\partial\phi_2}{\partial u}\right) + \left(D_1\frac{\partial\phi_2}{\partial v} - D_1'\frac{\partial\phi_2}{\partial u}\right)\left(D_1'\frac{\partial\phi_1}{\partial v} - D_1''\frac{\partial\phi_1}{\partial u}\right)}{2F_1}.$$

Hence, if any two surfaces are given which correspond by orthogonality of linear elements and a third surface can be found having the same kind of correspondence with either, and which is such that the characteristic functions of the latter satisfy these equations, the two original surfaces can be deformed in such a

way that the resulting surfaces will have the same kind of correspondence. This condition is satisfied if ϕ_2 is such that

$$D_1 \frac{\partial \phi_2}{\partial v} - D_1' \frac{\partial \phi_2}{\partial u} = 0, \quad D_1' \frac{\partial \phi_2}{\partial v} - D_1'' \frac{\partial \phi_2}{\partial u} = 0,$$

that is, excluding the case where S_1 is a developable surface,

$$\phi_2 = \text{const.}$$

As we have seen, this holds when S_1 and S_2 are adjoint minimal surfaces. Hence, if S_1 is a minimal surface and is deformed infinitesimally along directrices whose direction cosines are proportional to the point coordinates of its adjoint, the resulting surface corresponds by orthogonality of linear elements to the surface S'.

Suppose that for S_2 we take the surface S, then $\phi_1 = \phi_2$ and the ratios (9) reduce to

$$\frac{M_1^2}{E_1} = \frac{N_1^2}{G_1} = \frac{M_1 N_1}{F_1},$$

which has for common value zero. Hence, neglecting the case where S_1 is a developable surface, we have

$$\phi_1 = \text{const.};$$

that is, S and S_1 are minimal surfaces and S_1 is the adjoint of S or a homothetic surface of this adjoint. It follows, therefore, that the only case where the non-developable surfaces S and S_1 are such that the surfaces resulting from an infinitesimal deformation of them, defined by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \delta z_1,$$

and

$$x_1' = x_1 + \varepsilon_1 x, \quad y_1' = y_1 + \varepsilon_1 y, \quad z_1' = z_1 + \varepsilon_1 z,$$

correspond by orthogonality of linear elements, is that in which S is a minimal surface and S_1 its adjoint or a homothetic of the latter. In particular, if S and S_1 are adjoint minimal surfaces, the respective deformations giving this result are defined by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \varepsilon z_1,$$

and

$$x_1' = x_1 - \varepsilon x, \quad y_1' = y_1 - \varepsilon y, \quad z_1' = z_1 - \varepsilon z.$$

Another case is that in which the surfaces resulting from an infinitesimal deformation of S and S_0 correspond by parallelism of tangent planes. For this

discussion we make use of the following expressions for the direction cosines of the tangent planes to S' found by Genty:*

$$X' = X + \varepsilon(Yz_0 - Zy_0),$$
$$Y' = Y + \varepsilon(Zx_0 - Xz_0),$$
$$Z' = Z + \varepsilon(Xy_0 - Yx_0).$$

If the associate surface of S_0 in the given deformation is denoted by $\Sigma(\xi, \eta, \zeta)$, we have

$$X_0' = X_0 + \varepsilon_1(Y_0\zeta - Z_0\eta),$$

and similar expressions for Y_0' and Z_0'. But

$$X_0 = X, \quad Y_0 = Y, \quad Z_0 = Z,$$

hence, if the given correspondence holds between S' and S_0'', it is necessary that

$$\frac{x_0 - \tau\xi}{X_0} = \frac{y_0 - \tau\eta}{Y_0} = \frac{z_0 - \tau\zeta}{Z_0}, \tag{10}$$

where $\tau = \frac{\varepsilon_1}{\varepsilon}$.

When, in particular, we take $\varepsilon_1 = \varepsilon$, these equations become

$$\frac{x_0 - \xi}{X_0} = \frac{y_0 - \eta}{Y_0} = \frac{z_0 - \zeta}{Z_0}. \tag{11}$$

Since S_0 and Σ correspond by parallelism of tangent planes, it follows from (11) that they are parallel surfaces. Comparing equations (10) and (11), we remark that when $\varepsilon \neq \varepsilon_1$, Σ is a homothetic of a surface parallel to S_0. Since the converse of the above results is true, we have the following theorem:

The necessary and sufficient condition that S_0'', arising from an infinitesimal deformation of S_0, corresponds to S' by parallelism of tangent planes at corresponding points is that Σ be parallel to S_0, or a homothetic of a parallel.

Let Σ be a parallel of S_0 and denote by D_λ, D_λ', D_λ'' the coefficients of its second fundamental form. From (III, 11) we find, on remarking that S' and S_0'' have the same spherical representation, and that for any surface $HK = \mathcal{K}$,

$$\frac{D_0 - D_\lambda}{\mathcal{E}_0} = \frac{D_0' - D_\lambda'}{\mathcal{F}_0} = \frac{D_0'' - D_\lambda''}{\mathcal{G}_0}. \tag{12}$$

* Toulouse Annales, t. IX, E. 7.

The second fundamental coefficients of parallel surfaces are given by*

$$D_p = E_0 \frac{a}{\rho_1 \rho_2} + D_0 \left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$
$$D'_p = F_0 \frac{a}{\rho_1 \rho_2} + D'_0 \left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$
$$D''_p = G_0 \frac{a}{\rho_1 \rho_2} + D''_0 \left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$

where ρ_1 and ρ_2 are the principal radii of normal curvature of S_0. Substituting these values in (12), we have

$$\frac{E_0 - D_0(\rho_1 + \rho_2)}{\mathfrak{E}_0} = \frac{F_0 - D'_0(\rho_1 + \rho_2)}{\mathfrak{F}_0} = \frac{G_0 - D''_0(\rho_1 + \rho_2)}{\mathfrak{G}_0},$$

By making use of (11), we find that the common value of these ratios is $-\rho_1\rho_2$. Since these expressions involve the coefficients of only one surface, and since this surface is perfectly general, we have for any surface

$$E = D(\rho_1 + \rho_2) - \mathfrak{E}\rho_1\rho_2,$$
$$F = D'(\rho_1 + \rho_2) - \mathfrak{F}\rho_1\rho_2,$$
$$G = D''(\rho_1 + \rho_2) - \mathfrak{G}\rho_1\rho_2;$$

these reduce to formulæ (3), when S is a minimal surface.

PRINCETON UNIVERSITY, PRINCETON, N. J.

* Knoblauch, "Allgemeine Theorie der Krummen Flächen," p. 236.

Die Typen der linearen Complexe elliptischer Curven im R_r.

Von S. Kantor.

An einer ansehnlichen Zahl von Problemen und Theorieen der Geometrie und Functionentheorie erkannte man in den letzten Jahren, dass ihren Kern die Systeme rationaler und elliptischer Curven bilden und in den meisten Fällen führte dazu das Princip der Verminderung der adjungirten Functionen. Den Anfang machte, wie wohl nicht zu bezweifeln ist, meine Theorie der periodischen Transformationen (Neapel, 1884–1889 und Cr. J., Bd. CXIV), sofern man nicht das Schwarz'sche Theorem mit seinen Beweisen von Schwarz, Hettner, Nöther, Picard, Painlevé vorausstellen will. Dann kam Herrn Picard's übrigens durchaus selbständige Preisschrift, hierauf Castelnuovo's Anwendungen meines Principes der Verminderung der ϕ* auf die Curvensysteme auf Flächen (Math. Ann., Bd. XLIV, Società italiana, 1896, Accademia R. dei Lincei), das sich weiterhin auch F. Enriques angeeignet hat (Società italiana, 1896, und Math. Ann., Bd. XLIX). Im Jahre 1895 folgte meine (schon 1885 vollendete) Theorie der Transformationsgruppen in der Ebene (Berlin, Meyer u. Müller, 1895) und weiterhin kamen meine neuen Aequivalenztheorieen in den Monatsheften, 1899, nebst der Anwendung auf die rationale Zerlegung der birationalen Transformationen in der Ebene in Primfactoren. Auch das Gruppenproblem in R_3 und R_r, das nun an die Reihe kam, wird wie von zwei Säulen von zwei Theorieen gestützt, von denen die eine—über die Typen rationaler Curven—bereits veröffentlicht ist,† die zweite den Gegenstand der vorliegenden Arbeit bildet.

Die Methode dieser zweiten Arbeit ist von jener der ersten äusserlich verschieden, wenn auch im allerletzten Grunde ihr verwandt und ist aber auch unabhängig von allen wie immergearteten Vorarbeiten Anderer, und beruht einzig und allein auf den Principien, die ich in meinen Arbeiten in den Monatsheften für

* Ich weise auf seine ausdrückliche Feststellung meiner Priorität hin, die er in den Math. Ann., Bd. XLIV, p. 125–155 macht.

† American Journ. of Mathematics, vol. XXIII.

Math. und Physik, 1899,* eingeführt habe und auf einer Weiterentwickelung derselben. Aus diesem Grunde wird es wohl angebracht sein, wenn ich über die Entstehung jener beiden Arbeiten hier unter dem Texte einige chronologische Feststellungen mache.†

Für die allgemeinen Begriffe und Definitionen, die sich auf Curvencomplexe im R_r beziehen, kann ich auf meine Abhandlung verweisen, die im XXIII Bande dieses Journals abgedruckt ist.

§1.—*Ueber eine Classe von Hilfssätzen über ebene Curvensysteme.*

Sind einmal die Typentheoreme für rational distinct gegebene Systeme von

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* "Neue Äquivalenztheorie für die linearen Systeme rationaler, elliptischer und hyperelliptischer Curven in der Ebene." "Rationale Decomposition der birationalen Transformationen in ihre Primfactoren."

‡ Die erste Idee entstand mir im Winter 1895 aus der Untersuchung jener Transformationen und Gruppen von Transformationen des R_r, welche ein System von R_1 in sich transformiren, und daher i. a. eine Gruppe von Transformationen erzeugen. Es galt zu finden, wie weit man die Decomposition treiben könne. Hierzu erweisen sich Theoreme für die Erniedrigung der Ordnung einer solchen birationalen Transformation nothwendig, deren Fundamentalpunkte nicht sämmtlich rational gegeben sind und ich sah sofort, dass das Theorem von Nöther aus dem V. Bande der Math. Ann. durch ein complicirteres ersetzt werden müsse. Von hieraus entwickelten sich dann allmälig die weiteren Anschauungen über die rational gegebenen oder rational bestimmbaren linearen Curvensysteme (und, wie bei meiner damaligen Beschäftigung mit der r-dimensionalen Geometrie selbstverständlich war, auch über rational distinct bestimmbare M_{r-1}-Systeme im R_r), die ich in den genannten zwei Arbeiten niedergelegt habe. Im Winter 1896 stellte ich jene Bemerkung über die Nothwendigkeit eines mehr gegliederten Decompositionstheoremes und eine andere Entdeckung, die ich bis zur Stunde noch nicht veröffentlicht habe, in einer kurzen Note zusammen, von welcher 4 Mathematiker (die Herren Thomas Craig (†), Ferdinand Lindemann, Heinrich Weber, Giuseppe Jung) theils im December 1896, theils im Februar und März 1897 Abschriften in Händen hatten (und in einem Falle noch haben).

Im Laufe der Monate März und April 1897 gelang es mir dann, die von mir noch vermissten Beweise, die sich nun in den Monatsheften von 1899 publicirt finden und die absolut original sind, hinzuzufinden. Das Heft des 49. Bandes der Math. Ann., das eine Arbeit von Enriques enthält, die ebenfalls mit dem Irrationalen in der Geometrie sich beschäftigt und absichtlich einen prononcirt algebraischen Ursprung und Zweck hervorkehrt, lernte ich als privater Besucher der "Library of the National Observatory" in Washington (Ca.) nicht vor Ende Mai 1897 kennen. Auf dem Annalenhefte befand sich übrigens die Notiz: "Ausgegeben am 23. April, 1897." Um jene Zeit also waren meine Ideen über das Rationale in der Geometrie schon längst entwickelt und meine Beweise—übrigens für Gegenstände, auf welche die Enriques'sche Arbeit *gar keinen Bezug* hat—beinahe vollendet. Soviel über "*das Rationale in der Geometrie.*" Cf. meine Fussnote hier unten in §5.

Das Problem der vorliegenden Arbeit nun aber war bereits behandelt und zu einem Schlusstheorem, wenn auch einem sehr wenig detaillirten, fortgeführt in einer allerdings nur provisorischen, unvollkommenen und flüchtig gearbeiteten Skizze, die im December 1896 einem Mathematiker (Thomas Craig †) vorgelegen hat: "Abriss einer Theorie der endlichen Gruppen eindeutiger Transformationen im R_3", gleichwie die nunmehr bereits publicirte Lösung des Problemes über die Complexe rationaler Curven im R_3.

Curven $p = 1$ gegenüber rational distincten Transpositionen gegeben, so verbleibt noch die Nothwendigkeit, die gefundenen typischen Systeme *durch ihre sämmtlichen Transversalzahlen* einzeln zu characterisiren. Die gleiche Zusammenstellung wäre dann jedesmal erforderlich, wenn die Systeme elliptischer Curven auf zwei rational distinct gegebenen M_2 hinsichtlich ihrer Aequivalenz untersucht werden sollen.

Die auf die sämmtlichen typischen ebenen Systeme elliptischer Curven bezüglichen Theoreme hier vorzuführen, würde zu viel Raum erfordern; ich will als Vertreter nur einen Fall hervorheben, wobei ich gleich auf der M_2 handle:

"Ein lineares ∞^8-System elliptischer Curven C_n, das auf einer rational distinct abbildbaren M_2 rational gegeben ist, kann von einer der drei folgenden Arten sein:

"1. Art. Es besitzt eine rational distincte Transversalcurve einpunktigen Schnittes, ein rational distinctes lineares ∞^2-System rationaler dreipunktig schneidender Transversalcurven, das vom Range 4 ist und ein rational distinctes lineares ∞^1-System zweipunktiger Transversalcurven der C_n enthält. Es besitzt kein rational distinctes lineares ∞^3-System von Transversalcurven, welches auch die Transversalzahl s wäre, und kein rational distinctes ∞^7- oder ∞^8-System, wohl aber ein rational distinctes lineares System (Rang 5), welche $s = 7$ gegen die C_3 hat.

"2. Art. Es besitzt zwei rational distincte lineare ∞^1-Systeme von rationalen Transversalcurven mit $s = 2$, ein rational distinctes lineares ∞^3-System rationaler Transversalcurven (Rang 2) mit $s = 4$. Es besitzt zwei rational distincte lineare ∞^8-Systeme 1. Art. von Curven $p = 1$ mit $s = 10$, welche ein rational distinctes ∞^7-System mit $s = 8$ gemeinsam haben, welches letztere auch Bestandtheil des gegebenen Systemes ist.

"3. Art. Es besitzt ein rational distinctes (aber irrational gebundenes) Paar von ∞^1-Systemen rationaler Transversalcurven mit $s = 2$ gegen C_n, ein rational distinctes lineares ∞^3-System rationaler Transversalcurven mit $s = 4$, ein rationales Paar von ∞^8-Systemen 1. Art. von Curven $p = 1$ mit $s = 10$, und ihnen gemeinsam ein rational distinctes lineares System ∞^7 mit $s = 8$, welches Bestandtheil des gegebenen Systemes ist."

Aehnliche Theoreme gibt es für die ∞^9, ∞^7, ..., ∞^1 systeme. Aus diesen entnimmt man nun die folgenden Theoreme, die ich anführen will.

THEOREM I.—*Auf rational distinct auf eine Ebene abbildbaren M_2 kann es nur folgende von einander verschiedenen linearen Systeme elliptischer Curven geben,*

welche nicht rational distinct in einem höher dimensionalen linearen Systeme elliptischer Curven enthalten sind:

1. *das ∞^9-System*, 2. *das ∞^8-System der 2. Art*, 3. *das ∞^8-System 3. Art*, 4. *ein ∞^6-System*, 5. *ein ∞^5-System. Aus jedem anderen linearen in einer rational distinct abbildbaren M_2 rational distinct gegebenen Systeme von Curven $p = 1$ lässt sich, wenn auch nicht immer auf rational distincte Art, ein lineares Stammsystem einer der obengenannten 5 Arten ableiten, in dem es enthalten ist.*

THEOREM II.—*Diejenigen Untersysteme elliptischer Curven von jenen aus Theorem I., die nicht in einem rational distinct aus ihnen ableitbaren Stammsysteme des Theoremes I. enthalten sind, sind:*

1. *das ∞^6-System, das zum Typus die C_3 mit drei irrational verbundenen Punkten hat, so lange als diese drei Punkte nicht alineirt sind,*

2. *das ∞^3-System, das zum Typus die C_3 mit zwei rationalen Tripeln hat, so lange keines derselben alineirt ist, oder*

3. *das ∞^3-System mit rationalem Sextupel, das nicht in einer C_2 enthalten ist,*

4. *das ∞^3-System mit drei Punktepaaren, deren Verbindungslinien nicht alineirt sind und die nicht in einer C_2 enthalten sind.*

Von diesen Theoremen werden wir im §4 dieser Arbeit Gebrauch machen.*

§2.—*Ueber einige in dieser Arbeit häufig angewandte Methoden.*

Wir beschäftigen uns in den folgenden §§ mit der Untersuchung der innerhalb eines M_{r-1}-Systemes oder durch mehrere M_{r-1}-Systeme erzeugten Curvensysteme und Complexe. Dazu dienen uns die gleichzeitig erzeugten M_2-Complexe. Gerade in der in diesem § gegebenen Zusammenstellung wird man den Grundcharacter unserer Untersuchung hervortreten sehen: wir steigen nicht in den R_4 oder R_r auf, um den R_{r-1} zu beleuchten, sondern, um die complicirteren Verhältnisse im R_r zu erforschen, steigen wir herab in die uns von früher her bekannten niederen Räume und aber bis zum zweidimensionalen Gebiete. Wir könnten sogar bei Anwendung von Vorsicht bis in das eindimensionale Gebiet hinabgehen.

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*Eine Bemerkung mag hier am Platze sein. Man muss überhaupt die Typen der linearen Curvensysteme noch weiter dadurch in Unterfälle zerlegen, dass man z. B. bei C_3 drei Basispunkte in einer Graden oder 6 in einem Kegelschnitte voraussetzt, ebenso bei C_4 etwa 4 in einer Graden u. s. w. So können sogar unter den Basispunkten ganze Configurationen entstehen und dadurch typische Systeme, welche ihrerseits nicht äquivalent sind und doch den gemeinsamen Character eines der Obertypen an sich haben. Ich muss aber zur Beruhigung des Lesers bemerken, dass von jenen specificirten Transversaltheoremen keine Anwendung gemacht wird. Sie sind nur zur Beleuchtung des Principes erwahnt.

1. Hat man auf einer allgemeinen M_2 des gleichzeitig mit dem M_1-Complexe erzeugten M_2-Complexes das erzeugte Curvensystem $C_m^{(2)}$ derartig, dass sich *rational distinct* aus ihm ein höher dimensionales System $C_{m'}^{(2)}$ herleiten lässt, so muss auch das erzeugende M_{r-1}-System resp. müssen die erzeugenden M_{r-1}-Systeme so beschaffen sein, dass sie ohne Aenderung des Geschlechtes p der erzeugten Curven C durch Weglassung von Gebilden der Basisconfiguration (Punkte, M_1, M_2, ...) oder Erniedrigung von deren Vielfachheiten für die M_{r-1} in ein höher dimensionales System verwandelt werden, so dass der gegebene Curvencomplex in einem höher dimensionalen von Curven gleichen Geschlechtes "enthalten" ist.

Hiebei aber wird in den einzelnen Fällen wohl zu beachten sein, dass es nicht als rational distinct angesehen werden kann, wenn zu dem beschriebenen Vorgange die Hinzunahme anderer nicht in allen M_2 des Complexes *gleichmässig* erscheinenden Raumgebilde nothwendig ist.

Kann aber ferner auf M_2 ohne Aenderung des Ranges k im Systeme der $C_m^{(2)}$ dessen Dimension erhöht werden, so muss dies auch beim M_{r-1}-Systeme möglich sein und kann auf M_2 gleichzeitig k und p bei Erhöhung der Dimension constant erhalten werden, so muss dies auch bei den M_{r-1} der Fall sein.

Ist ein linearer Curvencomplex in keinem höher dimensionalen von gleichem p oder k oder p und k enthalten, so kann das aus ihm auf eine der erzeugten M_2 entfallende Curvensystem in keinem höher dimensionalen von gleichem p oder k oder p und k enthalten sein.

2. Das gleiche Princip gilt für den Uebergang von einem höher dimensionalen zu einem niederdimensionalen Systeme.

3. Ist auf einer allgemeinen M_2 des in einem Curvencomplexe $\infty^{u>r-1}$ des R_r erzeugten M_2-Complexes das erzeugte Curvensystem $C_m^{(2)}$ derartig, dass es ein rational distinctes Transversalensystem der Dimension u mit der Transversirungszahl s besitzt und dieses den Rang k_s hat, so gibt es auch ein *rational distinctes*, also aus dem erzeugenden M_{r-1}-Systeme oder den erzeugenden M_{r-1}-Systemen auf *rational distincte* Art ableitbares M_{r-1}-System, welches 1. gegen die sämmtlichen Curven des erzeugten Complexes die Transversalzahl s, 2. für sich i. a. die Dimension ∞^u und nur in Ausnahmsfällen eine grössere Dimension besitzt und 3. dessen in ihm erzeugte Curven die gegebenen M_{r-1} in k_s Punkten treffen (gegen sie die Transversalzahl k_s haben).

4. Eine Methode zur Construction von linearen Complexen elliptischer Curven: a) Hat man eine rational distincte Abbildung einer M_{r-1} auf die Ebene L_{r-1} mit dem Abbildungssysteme K_{r-2} und kann man im L_{r-1} ein ∞^u-System P_u elliptischer Curven verzeichnen, sodass es ein Restsystem im Systeme $(K_{r-2})^r$ hat und dass diese Ergänzungen so gewählt werden können, dass auch das in $C_1 + P_u$ schneidende System von M_{r-1} lauter abbildbare M_{r-1} enthält, was, wenn u hinreichend gross, bei elliptischen P_u übrigens eine nothwendige Folge ist, und dass die C_1 auf diesen M_{r-1} dieselben Singularitäten wie auf der gegebenen M_{r-1} bewirken, so hat man ein diese M_{r-1} enthaltendes M_{r-1}-System, das einen Complex elliptischer Curven erzeugt.

In der Anwendung dieser Methode, namentlich wenn sie mit Princip No. 1 in Verbindung gesetzt wird, ist vorsichtig zu beachten, dass durch die Abbildung ein neues Moment in das System elliptischer Curven gebracht wird: *irrationale Constructionselemente.* Man darf also nicht daraus, dass nach vollzogener irrationaler Abbildung der Bildcomplex in einem höher dimensionalen Complexe (etwa sogar distinct) enthalten ist, sofort schliessen, dass auch der Originalcomplex in einem höher dimensionalen Complexe enthalten sein müsse. Man muss eben beachten, dass das, was ich in No. 1. bis 3. dieses § als "die allgemeine M_2" im Complexe bezeichnet habe, noch nicht durch die Abbildung allein bestimmt ist, sondern erst, wenn man ausserdem noch die gemeinsamen Elemente des M_2-Complexes bestimmt hat. Erst durch die Kenntniss dieser kann man auch auf der einzelnen M_2 den Character des Curvensystemes puncto Transversalen in Bezug auf den ganzen Complex bestimmen.

b). Im Falle, wo die M_{r-1} rational distinct auf einen L_{r-1} abbildbar ist, gilt: Nimmt man in der Methode a) für P_u zwei geometrisch äquivalente, also durch räumlich birationale Verwandtschaft unter einander übergehende Systeme, sofern man die beiden linearen Systeme so situirt, dass die Transversalzahlen des Curvensystemes im Complexe dieselben bleiben, so ist auch der eine durch die Construction entstehende Curvencomplex im R_r wirklich geometrisch äquivalent dem erzeugten Curvencomplexe bei der zweiten Annahme. Es muss aber jene Verwandtschaft im L_{r-1} auf rein rationalem Wege für jede M_2 des Systemes erreichbar sein.

Es kann sonst geschehen, dass, obzwar die beiden projicirten Complexe elliptischer Curven vollständig übereinstimmen, die beiden Curvencomplexe dennoch nicht äquivalent sind. Von diesen Erwägungen werden wir besonders in §4 Verwendung machen.

c). Im selben Falle wie b) gilt: Nimmt man in der Methode a) einen Complex P_u nur in verschiedenen Lagen gegen K_{r-2}, so erhält man durch die Construction zwei Complexe elliptischer Curven im R_r, welche wirklich geometrisch birational unter einander übertragbar sind.

d). Mit derselben Beschränkung wie b) gilt: Man erhält durch die Methode aus a) räumlich birational verwandte Curvencomplexe, wenn man den Complex P_u im Abbildungsraume L_{r-1} ungeändert lässt, aber das Abbildungssystem K_{r-2} durch irgend ein anderes ersetzt, welches noch immer ein ∞^{u+1}-System in R_r liefert, und wenn aber überdies nicht nur in der Abbildung, sondern auch auf "der allgemeinen M_2 des Complexes" ein System genau derselben Art wie beim anderen entsteht.

5. Die Methode aus No. 4 mit den nothwendigen Cautelen bleibt in Giltigkeit, wenn es sich um die Construction von Curvencomplexen und deren birationaler Aequivalenz *innerhalb* einer abbildbaren M_{r-1} oder zweier geometrisch birational bezogenen abbildbaren M_{r-1} handelt, von welchen man eine rational distincte Abbildung auf einander besitzt. Hiebei aber darf die Aequivalenz *auf* den M_{r-1}, also durch Correspondenzen unter den Punkten der beiden M_{r-1} allein nicht mit der Aequivalenz *in* den umgebenden R_r verwechselt werden.

§3.—*Eine umfassende Classe uneigentlicher Complexe elliptischer Curven im R_3.*

Auf einer M_2 sei ein lineares ∞^u-System von Curven $p=1$ gegeben, $u \geqq 3$. Es gibt unter den ausschneidenden M_{r-1} solche, welche in einem willkürlichen Punkte P von M_2 berühren. Ist $u=3$, worauf sich zu beschränken genügt, so haben wir ∞^2 Curven mit Doppelpunkt. Diese sind rational, geben also, wenn man P längs einer dieser Curven selbst variiren lässt, sicher unter anderem auch ein rationales ∞^1-System rationalen Curven, demzufolge M_2 sofort nach einem Nötherschen Satze unicursal wird, oder aber zerfallen in zwei Bestandtheile, die sich in P kreuzen und nicht beide elliptisch sind, da die Summe zweier elliptischer Curven nur dann auch eine elliptische Curve ist, wenn jene einem (durch die Singularitäten bestimmten) Büschel von Curven $p=1$ angehören, was auch dann gilt, wenn die beiden Curven sich irgendwie im Raume befinden; von den Bestandtheilen ist also mindestens einer rational, sodass sich ∞^1 oder ∞^2 rationale Curven auf M_2 befinden, was im zweiten Falle die M_2 unicursal macht. Der erste Fall kann die ∞^1 Curven $p=0$ nur in einer Reihe vom Index 1 gesammelt haben. Denn würden durch P zwei Curven

gehen, so könnte man, da auch die zweiten Bestandtheile der Zerfallcurven eine Reihe bilden, mehr als eine Curve der ∞^3-Reihe construiren, welche in P einen Doppelpunkt hätten, somit lineare ∞^1-Reihen, welche M_2 als unicursal bedingen. Also:* *Jede M_2, auf der eine lineare ∞^3-Reihe elliptischer Curven ausgeschnitten werden kann, ist entweder unicursal oder enthält ein Büschel rationaler Curven vom Index* 1, *hat also das Geschlecht $p < 0$† und ist vom Typus der elliptischen Kegelfläche.*

Auf einer unicursalen M_2 kann kein lineares System elliptischer Curven eine Dimension > 9 haben, auf den Kegelflächen $p = -1$ kann die Dimension eines linearen Systemes elliptischer Curven unbeschränkt hoch sein.

Dass die M_2 2. Art vom Typus der Kegelflächen sind, sieht man daraus, dass man das ∞^3-System in ein System ebener Schnitte verwandeln kann und auf der neuen M_2 können dann die rationalen Bestandtheile, die oben gefunden wurden, nur Geraden sein.

Ist nun ein *lineares System vom M_2 der zweiten Art* vorhanden, so müssen diese Büschel sämmtlich in einem Complexe rationaler Curven enthalten sein, von dem durch jeden Punkt des R_r nur eine geht. Die M_2 eines solchen Systemes schneiden sich also in Curven, welche aus mehreren Curven dieses Complexes zusammengesetzt sind. Daher können die M_2 vom Typus der elliptischen Kegelflächen nicht zu eigentlichen Complexen Anlass geben und es gilt:

THEOREM III.—*Alle uneigentlichen Complexe elliptischer Curven im R_3, von denen ein erzeugendes M_2-System den Typus der elliptischen Kegelflächen besitzt, sind räumlich birational übertragbar in den Schnitt eines linearen Systemes elliptischer Kegeln mit einem irgendwie hoch dimensionalen Systeme von Monoiden.*

Denn auf elliptischen Kegeln werden die elliptischen Curven von Monoiden ausgeschnitten.

Anmerkung. Es ist selbstverständlich, dass in einer Abhandlung über *Complexe von elliptischen Curven* (und besonders von so *weiten Zielen wie diese*) die Mannigfaltigkeiten mit elliptischen Curven, also auch die M_2 mit Systemen solcher vorkommen *müssen*. Ich habe daher den eben vorhergegangenen Beweis für M_2 mit ∞^3-Systemen gegeben und will nun zeigen, wie sich *aus jenem Satze*

* Dass jede M_2, auf der ein lineares ∞^2-System elliptischer Curven vorhanden ist, unicursal oder vom Kegelflächentypus sein müsse, hat G. Castelnuovo zuerst bewiesen in Rend. Acc. Lincei 1894.

† Auch dieser Satz, den ich in den C. R. vom 12. November, 1900, bewiesen, nachdem ihn vorher Herr F. Enriques, 1899, in den Math. Annalen ausgesprochen und bewiesen hatte, war bereits in jenem "Abrisse" von 1896, den ich oben erwähnte, enthalten und der später von mir in den C. R. gegebene Beweis daselbst angedeutet.

der Castelnuovo'sche Satz über M_2 mit ∞^2-Systemen ableiten lasse, und das mittelst einer Entwickelung, die *nichts* von den bekannten Beweisen an sich hat:— Ist auf der M_2 ein ∞^2-System enthalten, so kann dieses, wenn es vollständig sein soll, keinen Rang > 2 haben. Denn da von den Schnittpunktegruppen auf den einzelnen Curven nur ein Punkt abhängig von den übrigen ist, so können nicht im Netze alle bis auf einen abhängig sein. Das Netz müsste also in einem auf der M_2 ausschneidbaren Systeme höherer Dimension enthalten sein, wo nur 1 Punkt abhängig ist und die Curven wieder elliptisch wären. Die M_2 mit ∞^2-Systeme vom Range 2 kann dann sofort auf eine M_2^n mit $(n-2)$-fachem Punkte O ein-eindeutig abgebildet werden. Welche diese M_2^n sind, wo die Ebenen durch O in elliptischen Curven schneiden, soll hier nicht erforscht werden, aber man weiss, dass das Netz durch die Uebergangscurve $p = 3$, welche alle C_3 in Punktquadrupeln schneidet, vollkommen bestimmt ist, was die Vertheilung der Invarianten betrifft. Man kann also zu jedem Netze auf einer M_2 eine Curve $C_4\ p = 3$ in einer Ebene construiren, welche, wenn sie auf jene Uebergangscurve in $M_2\,(1-1)$ deutig bezogen ist, auch dieselbe Vertheilung der absoluten Invarianten unter den C_3 des Netzes in der Ebene bewirkt, wie unter den elliptischen Curven auf M_2 und es also ermöglicht, dass unter je zwei durch jene Beziehung einander zugewiesenen elliptischen Curven eine (1, 1)-deutige Beziehung hergestellt werden könne, sodass diese insgesammt eine (1, 1)-deutige Beziehung unter der M_2 und der Ebene liefern.

Hieraus schliessen wir auch: *Ist auf einer M_2 ein lineares ∞^2-System elliptischer Curven vorhanden, so ist dieses in einem ∞^3-System (eventuell nicht rational distinct) enthalten und M_2 ist* (1, 1)*-deutig auf eine Ebene abbildbar oder vom Typus der elliptischen Kegelflächen.*

§4.—*Die eigentlichen linearen Complexe elliptischer Curven der I. Classe im R_3.*

1. Die Natur des Complexes hängt offenbar von der Natur des linearen Systemes ab, das aus dem Curvencomplexe auf eine im Complexe erzeugte M_2 entfällt. Wir werden sehen, dass nach der Art dieser Systeme die Complexe in Classen einzutheilen sind und wollen diese System kurz die zweifach ausgedehnten Systeme im Complexe nennen.

In die I. Classe sollen nun jene Complexe gerechnet werden, wo die M_2 rational distinct auf eine Ebene abbildbar ist, nämlich so wie sie allgemein im Complexe erscheint. In diesem § sollen alle Complexe dieser Art erledigt werden. Vor allem gilt der ganz evidente Satz:

"*Soll die allgemeine M_2 des Complexes rational distinct auf die Ebene abbildbar sein, so muss es auf der M_2 stets ein rational distinguirbares System ∞^2 Ranges 1 rationaler Curven geben.*" Denn in der rational gegebenen Ebene, in die die M_2 distinct verwandelbar sein soll, gibt es stets ein unbedingt rational distinctes vollständiges ∞^2-System rationaler Curven, nämlich das Geradensystem.

2. Es sei nun im R_3 ein ∞^u-System von M_2 gegeben, das einen Complex von C_n $p=1$ erzeuge und die M_2 seien von der besprochenen Art. In jeder M_2 entsteht dann ein ∞^{u-1}-System, dessen Natur nun mit der irgend eines der in der Ebene möglichen Typen elliptischer Curvensysteme übereinstimmen muss. Von diesen Typen ist zu Beginn des §1 die Rede gewesen. Jedoch können nicht einmal alle jene Typen hier entstehen, weil das auf der M_2 vorhandene rational distincte ∞^2-System von Curven $p=0$ auch rational distinct für das lineare System elliptischer Curven sein muss, womit aber nicht gesagt ist, dass es sich aus diesem müsse auf rational distincte Art herleiten lassen.

Nach No. 1 des §2 schliessen wir nun, dass für das System von M_2 ein lineares System von Transversal-M_2^1 existiren muss, welches jene M_2 in jenem betrachteten rational distincten ∞^2-Systeme $p=0$ schneidet. Diese M_2^1 werden sich gegenseitig in Curven Γ schneiden, die gegen die M_2 des gegebenen ∞^u-Systemes die Transversalzahl besitzen und welche also, da die M_2 ∞^u sein sollen, es also ein ∞^u-System gibt, welches die einzelnen Punkte der Γ ausschneidet, rational sein müssen. Wir haben jetzt ein vollständiges ∞^2-System von M_2^1 mit rationalen Schnittcurven, woraus nebenher nach dem bekannten Satze Nöther's folgt, dass die M_2^1 auch unicursal sind. Wesentlich ist aber: im Besitze eines linearen ∞^2-Complexes rationaler Curven zu sein. Beziehen wir diesen Complex auf die ∞^2-Strahlen eines Bündels und dann überdies ein Büschel von M_2 des gegebenen Systemes auf ein Büschel von Ebenen, so erhalten wir die Möglichkeit, eine birationale Transformation einzurichten, welche jenen ∞^2-Complex in ein Strahlbündel, also die M_2^1 in ein Ebenenbündel und die M_2 in ein monoidales ∞^u-System verwandelt. Also:

THEOREM IV.—*Jedes ∞^u-System der I. Classe ist räumlich birational äquivalent einem ∞^u-Systeme von Monoiden mit gemeinsamem Scheitel M_2^n (O^{n-1}).*

3. Wir können im gegenwärtigen Falle das Princip No. 4, b) aus §2 einfach beweisen, indem wir für zwei gegebene Monoidsysteme die Schnittcurven auf je einem Monoide von O_1, O_2 aus herabprojiciren, so zwei ∞^{u-1}-Curvensysteme erhalten, welche birational äquivalent sind, zwei in dieser Aequivalenz A ent-

sprechende Curven benützen, hiezu zwei Büschel von Monoiden construiren und eine Raumtransformation mittelst A (unter den Strahlbündeln) und einer Projectivität auf entsprechenden Strahlenpaaren einrichten, und überzeugen uns, dass:

Zwei ∞^u-Systeme von Monoiden $M_2^{n_1}(O_1^{n_1-1})$, $M_2^{n_2}(O_2^{n_2-1})$ mit elliptischen Schnittcurvecomplexen birational äquivalent sind, wenn nicht nur die auf je einem Monoide enstehenden Curvensysteme von O_1, O_2 aus in zwei Systeme der gleichen Typus aus §1 projicirt werden, sondern überdies die Rationalitätsverhältnisse der M_2 innerhalb des Complexes dieselben bleiben.

4. Unser Problem hat nunmehr folgende Form zu bekommen: Wir müssen die Typen des §1 einzeln durchgehen, indem wir sie als Projection eines in einem Monoide $M_2^n(O^{n-1})$ enthaltenen Systemes ansehen, auf dem es aber durch ein lineares System von Monoiden derselben Ordnung ausgeschnitten wird und müssen, indem wir von den niedersten Werthen des n beginnen, zu erforschen suchen, welche irrational distinguirte Systeme auf $M_2^n(O^{n-1})$ flächen wir mit demselben §1-Typus erreichen können. Dies soll nun geschehen.

$$C_3\text{-}Typus\,(A_1, \ldots .).$$

5. $M_2^3(O)$. Sie sollen durch die Abbildung untersucht werden, welche den ebenen Schnitten die C_2 durch zwei feste Punkte O_1, O_2, die man als ein irrational verknüpftes Paar ansehen muss, entsprechen machen. C_3 als Bilder von Schnitten solcher M_2^3, welche O enthalten (damit sie in diesen § gehören), können nur insofern erreicht werden, als sie durch O_1O_2 gehen. Wir haben also hier als Bild ein ∞^7-System, welches nicht in einem ∞^9-Systeme, sondern nur in einem ∞^8-Systeme (als Zerfällungsproduct) enthalten ist, wenigstens mit Beachtung dessen, dass $M_2(O)$ in einem linearen Systeme enthalten sein soll. Für uns soll der so entstehende Complex der Schnittcurven aller ∞^8-$M_2^3(O)$ als Typus gelten. Wir erhalten aus ihm andere ∞^u-Systeme, $u < 0$, wenn wir weitere einfache Punkte als gemeinsam voraussetzen, $A_2, \ldots, A_7$, wo aber die Verknüpfungen wie $\overset{\frown}{A_2A_3}$, $\overset{\frown}{A_2A_3A_4}$, $\overset{\frown}{A_2A_3A_4A_5}$, $\overset{\frown}{A_2A_3A_4A_5A_6}$, $\overset{\frown}{A_2 \ldots A_7}$* zu beachten sind und mit rational gegebenen Punkten in allen dort angegebenen Arten zu combiniren sind.

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* Das Zeichen $\frown$ meint, dass die darunter zusammengefassten Punkte irrational sind, aber insgesammt eine rationale Gruppe bilden.

6. $M_2^3(O^2)$. Die Abbildung der ebenen Schnitte wird durch C_3 durch 6 Punkte $O_1, \ldots, O_6$ geliefert, die in einem Kegelschnitte Ω_2 sind. Aller $M_2^3(O^2)$ gegenseitige Schnitte haben als Bilder C_5 und die auf der einen $M_2^3(O^2)$ die $C_5(O_1, \ldots, O_6)$. Hier kann also ein ∞^9-System C_2 erscheinen, das aber durch Ω_2 zu ergänzen ist. Also haben wir als zweiten Typus den Complex der Schnittcurven aller $M_2^3(O^2)$ mit festem Osculationskegel in O. Da die C_3 keine Basispunkte gemeinsam haben, so kann sich in den Transversalzahlen nichts ändern, wenn wir nach §2, No. 4, eine andere Abbildung nehmen und wir können sofort diesen Typus als den einzigen der ∞^{10}-M_2-Systeme erklären.

Beim ∞^8-Systeme mit 1 Punkte A_1 können wir A_1 willkürlich in der Ebene nehmen oder mit O_1 coincidiren lassen, das heisst aus dem vorigen M_2^3-Systeme eines durch Festlegung eines Punktes ausscheiden, und diesen aber entweder willkürlich im Raume oder auf einer Erzeugenden des osculirenden Kegels wählen. Die beiden so entstehenden Systeme sind äquivalent. Wir sehen, dass in beiden Fällen der Ergänzungskegelschnitt C_2 mit Ω_2 coincidiren muss.

Beim ∞^7-System mit 2 Punkten A_1, A_2 gilt derselbe Schluss wie vorhin, nämlich dass bei gleicher Wahl von C_2, nämlich $\equiv \Omega_2$, die Verlegung von A_1, A_2 frei in die Ebene oder in O_1, O_2 keine birational verschiedenen Systeme liefert. Bei der zweiten Wahl sind wir *nicht gezwungen, $C_2 \equiv \Omega_2$ zu nehmen*, sondern können C_2 frei durch $O_3\, O_4\, O_5\, O_6$ wählen. Dann entsteht ein von den beiden anderen nicht verschiedenes System $M_2^3(O^2)$, welches einen festen Kegelschnitt Q besitzt, der O nicht enthält. Eine quadratische Transf. $M_2^2(OQ)$ führt aber dieses System sofort auf $M_2^2(O)$, den 1. Typus.

Wir haben in dem hier erhaltenen ∞^8-System von $M_2^3(O^2)$ mit zwei Geraden $g_1 g_2$ durch O und festen Osculationskegel in O einen Beleg für die in No. 4 des §2 angerathene Vorsicht. Das C_3-System mit festem Punktepaare als Bild liefert bei $M_2^2(O)$ und bei $M_2^3(O^2)$ zwei ganz verschiedene Complexe.

Beim ∞^6-System von C_3 mit $A_1 A_2 A_3$ können wir die $A_1 A_2 A_3$ frei oder theilweise oder ganz in O_1, O_2, O_3 wählen. Ein System, das nicht aus den bisherigen durch Festlegung eines weiteren gemeinsamen Punktes, der ohne Unterscheidung auch auf einer Erzeugenden des Osculationskegels gewählt werden kann, hervorgeht, entsteht durch $A_i \equiv O_i$ $(i = 1, 2, 3)$ und wenn man den Ergänzungskegelschnitt frei durch $O_4\, O_5\, O_6$ legt. Dies liefert $M_2^3(O^2)$ durch eine feste cubische Raumcurve, welche O enthält. Wir haben also hier zwei verschiedene Typen von ∞^7-M_2^3-Systemen.

Beim ∞^5-System können wir $A_1, \ldots, A_4$ mit $O_1, \ldots, O_4$ coincidiren lassen, was $M_2^3(O^2)$ mit festem Osculationskegel und 4 Geraden durch O liefert, falls wir den C_2 mit Ω_2 identisch nehmen. Dazu sind wir nicht gezwungen; legen wir also C_2 frei durch $O_5\ O_6$, so haben wir ein ∞^6-M_2^3-System mit O^2 und einer festen Curve $C_4(O^2)$. Dieses System ist ersichtlich mit dem anderen nicht äquivalent, wie ein Vergleich der Transversalzahlen beweist, es ist aber ebensowenig mit einem M_2^2-System äquivalent, und da wir andere noch nicht untersucht haben, mögen wir es als einen neuen Typus erklären.

Beim ∞^4-System von C_3 erhalten wir ebenso nur ein System, eines enthalten im ∞^{10}-Systeme von $M_2^3(O^2)$ von vorhin, das andere entstehend durch einen Ergänzungskegelschnitt durch A_6, also ein $M_2^3(O^2)$-System mit fester Curve $C_5(O^3)$. Auch dieses System ist auf keines der vorigen reductibel.

Es ist zu verzeichnen, dass jedes dieser beiden Systeme ∞^6, ∞^7 (mit $C_4(O^2)$ und $C_5(O^3)$) sofort reductibel wird, wenn man die C_4, C_5 zerfallen lässt, aber nur insofern, als ein System entsteht, das Untersystem eines der bisherigen Typen, also "unvollständig" ist.

Beim ∞^3-System von C_3 müssen wir nun beachten, ob die 6 gemeinsamen Punkte des Bildes sollen in einem Kegelschnitte sein oder nicht. Im ersten Falle haben wir die nicht unterscheidende Möglichkeit, die Punkte $A_1, \ldots, A_6$ aus der freien Ebene heraus auf die $O_1, \ldots, O_6$ zu verlegen und erhalten ein Untersystem unseres ∞^{10}-Systemes von $M_2^3(O^2)$. Im zweiten Falle können wir, falls 5 Punkte sich rational von dem Sextupel abtrennen lassen, diese 5 auf $O_1, \ldots, O_5$ verlegen, müssen aber A_6 ausserhalb lassen und erhalten also ein Untersystem des vorigen $M_2^3(O^2)$ mit $C_5(O^3)$. Sind aber $\overbrace{A_1, \ldots, A_6}$ irrational verbunden, dann müssen wir sie frei in der Ebene wählen und erhalten nothwendig $C_2 \equiv \Omega_2$, also wieder ein Untersystem des ∞^{10}-Systemes. Im ersten Falle bleibt jedoch möglich, C_2 anders zu wählen und das, um kein Untersystem eines bekannten zu finden, eben so, dass wir C_2 ganz frei in der Ebene wählen. Dies liefert $M_2^3(O^2)$ mit fester $C_6(O^4)$. Auch dieses System ist nicht identisch mit den schon gefundenen Typen.

Das ∞^2-System gibt nun nothwendig ein Untersystem eines der bisherigen, das also durch Festlegung eines einfachen Raumpunktes entsteht, der aber auch so genommen werden kann, dass sich eine weitere gemeinsame Linie des $M_2^3(O^2)$-Systemes ergibt.

Diejenigen ∞^1-Systeme von C_3, welche zu $M_2^3(O^2)$-Systemen führen, die

nicht Untersysteme der vorigen Systeme sind, mögen hier nicht aufgezählt werden.

7. Aber es fehlen uns noch jene Systeme, welche alineirte Tripel enthalten. Da haben wir bei den $M_2^2(O)$ bei 3 bis 7 Punkten A_i des Bildes bez. 1, 1, 2, 4, 6 Alineationen als mögliches Maximum und dem entspricht, dass O mit 1, 1, 2, 4, 6 Tripeln (maxime) unter den übrigen 3 bis 7 Basispunkten complan ist.

8. Uebergehend zu den $M_2^3(O^2)$ können wir dort, wo mehr als drei A_i im Bilde ausserhalb der O_i sind, die Alineationen sofort eintragen; ebenso wenn wir in allen Fällen als Typen jene Systeme $M_2^3(O^2)$ gewählt hätten, wenn wir die A_i frei in die Ebene anstatt auf die O_i verlegen. Es ist nun aber unmöglich, sie dort einzuführen, wo die C_2, Ω_2 verschieden sind, also verschiedene A_i nothwendig auf die O_i verlegt werden müssen. Unter diesen $A_i \equiv O_i$ können nur dann Alineationen supponirt werden, wenn Ω_2 selbst in zwei Geraden zerfällt.

9. Wir sind damit zur Nothwendigkeit geführt, solche Systeme von $M_2^3(O^2)$ zu untersuchen, wo alle Osculationskegel zerfallen, denn würde ein conischer Punkt in O erscheinen, so könnte diese M_2^3 zur Abbildungsdiscussion benutzt werden. Ein solches System ist nur möglich 1. wenn eine der beiden Osculationsebenen in O fest ist, 2. wenn alle Ebenenpaare in einem Büschel enthalten sind. Wir discutiren zunächst den ersten Fall.

Sofort erhalten wir, indem wir auf der festen der beiden Geraden ω_1, ω_2, in die Ω_2 zerfällt, die drei Coincidenzpunkte $A_i \equiv O^i$ nehmen müssen, ein ∞^7-System von $M_2^3(O^2)$ mit biplanarem Doppelpunkte, einer festen Osculationsebene $O\omega_1$ und einer $C_3(O^2)$ und ein damit nicht äquivalentes ∞^7-System mit zwei festen Osculationsebenen in O und drei festen einfachen Geraden in einer derselben. Auch die $C_3(O^2)$ kann in $C_1(O) + C_2(O)$ oder 3 Geraden in $O\omega_1$ zerfallen.

Niederdimensionale Systeme, welche nicht in diesen beiden enthalten sind und gleichzeitig nicht äquivalent mit den bisherigen Typen wären, erscheinen nicht. Bei ∞^9 $M_2^3(O^2)$ ist es kein Unterschied, ob der feste Osculationskegel zerfallend ist, weil dadurch nur eine sonst nicht distincte 2-punktige Transversale eintritt; analog beim ∞^8-System. Auch das System: ∞^4 $M_2^3(O^2)$ mit Uniplanarität in O nebst fester Osculationsebene $O\omega_1$, sowie drei festen Geraden darin und festen Berührungsebenen längs dieser ist typisch nichts neues.

Ebenso ist das ∞^{10}-S. mit zerfallendem festem Osculationskegel äquivalent dem mit conischem Doppelpunkte.

Auch ist es bei festem Osculationsebenenpaar $O\omega_1$, $O\omega_2$ keine wesentliche Unterscheidung in Bezug auf die räumliche Aequivalenz, wenn man den Doppelpunkt von Ω_2 als Träger von zwei Punkten O_i nimmt (§ 2).

Sind in den einzelnen bisher behandelten Fällen mit Alineationen noch einfache Punkte A_i frei in der Ebene möglich, so können nicht äquivalente Systeme noch dadurch entstehen, dass unter diesen für sich oder unter ihnen mit Bezug auf die Punkte O_i Alineationen entstehen. Sie gehören in die aufzustellende Tabelle.

Bleibt der 2. Fall. Da die $O\omega_1$, $O\omega_2$ ein Büschel bilden, so sind alle Dispositionen der A_i mit festem Ω_2 ausgeschlossen. Es können aber, wie eine Discussion zeigt, weder mit $C_3(O)$, noch $C_4(O^2)$, $C_5(O^3)$, $C_6(O^4)$ Systeme von biplanaren Monoiden entstehen, solange die 6 Punkte O_i von den Punkten A theilweise verschieden sind. Es gibt aber eine Möglichkeit der Annahme, bei der keine festen Tangenten der Basiscurve in O entstehen, welche die Variabilität der ω stören, und ebenso keine O_i-Durchgänge für die $O\omega$; das ist, wenn C_2 ein Geradenpaar ist, dessen Scheitel mit dem des Geradenpaares ω_1, ω_2 coincidirt. Dies liefert $\infty^4 M_2^3(O^2)$ mit zwei festen Ebenen C_3, die in O eine gemeinsame Spitze mit gemeinsamer Spitzentangente haben.

$M_2^4(O^3)$. Das System K (§ 2, No. 4) bilden $\infty^3 C_4$ durch zwölf O_i in einer Cubik Ω_3. Die $M_2^4(O^3)$ schneiden in Curven, deren Bilder C_7 durch die O_i sind. Hier wie bei den M_2^5 u. s. w. existirt für das ∞^3-System C_3 ein Ergänzungs-C_4 durch die O_i, welche aber keine wesentlich neue Transversal-M_2 oder-M_1 im R_3 bedingt Alle so erhaltenen Complexe (in ∞^{10}-Systemen von Monoiden) sind birational äquivalent. Dasselbe gilt für die durch Annahme einfacher Basispunkte A. abzusondernden Untersysteme. Eine zweite Möglichkeit entsteht erst bei 3 alineirten Punkten A_i. Für $C_3 \equiv \Omega_3$ als Bestandtheil von C_4 entsteht nichts neues geht die C_4 durch 10, 9, 8, 7 Punkte O_i, während die übrigen O_i Punkte A_i für die C_3 sein sollen, entstehen M_2^4-Systeme, welche mit schon gefundenen $M_2^3(O^2)$-Systemen mit Basiscurve äquivalent sind. Jene Möglichkeit tritt beim ∞^3-System C_2 ein, wenn die 6 Punkte A_i nicht conconisch sind. Bei den $M_2^3(O^2)$ konnten wir dann die sechs A_i nicht mit O_i coincidiren lassen und wegen der Möglichkeit irrationaler Verknüpfung mussten wir Typen zulassen, wo die sechs A_i frei in der Ebene waren. Hier können wir in jedem Falle eine Annahme

machen, wo sechs $A_i \equiv O_i$ sind. C_4 enthält die übrigen sechs A_i und es entsteht ein neuer Typus, nämlich ein ∞^4-System von $M_2^4(O^3)$, mit fester $C_{10}(O^6)$.

Selbe Möglichkeit für 10 A_i und, wollten wir gleich auch ∞^1-Systeme schaffen, auch für 9 A_i. Es entsteht ein neuer Typus: $\infty^3 M_2^4(O^3)$ durch eine feste $C_{11}(O^7)$.

Es bleibt übrig, zu untersuchen, ob die C_3-Systeme mit Alineationen neue Typen liefern. Hier sind alle Combinationen von Alineationen unter 3 bis 7 Punkten O_i möglich, ohne das deswegen die Ω_3 zerfallen müsste. Es ist aber zu beachten, dass, wenn diese O_i nicht feste Geraden abbilden, dann erst durch die Lage der übrigen Punkte O_i zu sorgen ist, damit auf jeder $M_2^4(O^3)$ jene Geraden, welche die Complexcurven 1-punktig schneiden, diese besondere durch die Alineationen gekennzeichnete Lage besitzen, was aber für mehr als drei O_i nicht möglich ist. Für drei $O_i \equiv A_i$ erhalten wir das ∞^7-System, dessen Typus bei $M_2^3(O^2)$ gefunden wurde. Für die Stammsysteme der übrigen Alineationen müssen wir entweder feste Geraden durch O oder feste einfache Basispunkte verwenden.

Bei 7 Punkten A_i ist noch Combination von Alineationen mit conconischen Lagen möglich.

Da wir für alle verschiedenen C_3-Systeme (cf. §1 und Monatshefte, 1899) und alle irrational distincten Lagen derselben gegen die O_i je ein M_2^2, M_2^3, M_2^4-System erhielten, müssen wir über die M_2^4-Systeme gar nicht hinausgehen.

$$C_4\text{-}Typus\ (A_1^2 A_2^2 A_3 \ldots).$$

$M_2^2(O)$ gestattet, sofern wir bei der Classe dieses § bleiben, kein System dieses Typus, weil bei gemeinsamem O die Bilder sofort C_3 werden.

$M_2^3(O^2)$. Bilder der Complexcurven waren die $C_5(O_1, \ldots, O_6)$. C_4 erfordern als Ergänzungscurve eine Gerade. Daher können wir $\infty^8 C_4$ nicht situiren, auch nicht mit $A_1^2 \equiv O_1$, $A_2^2 \equiv O_2$, solange nicht Ω_2 zerfällt. Aber auch dann noch ist das ∞^8-System $M_2^3(O^2)$ mit fester Osculationsebene in O und zwei nicht in dieser Ebene enthaltenen Geraden durch O nicht typisch. Dagegen müssen wir allerdings für $C_4(\overbrace{A_1^2 A_2^2}\ \overbrace{A_3 A_4})$ ein System $M_2^3(O^2)$ mit zwei Geraden $g_1 g_2$ durch O und einer Geraden fremd zu O als typisch erklären, wenn wir am Rationalen festhalten. $g_1 g_2$ müssen sogar nicht irrational verknüpft sein.

Das ∞^5-System $M_2^3(O^2)$ mit ebener $C_3(O^2)$ und zwei auch nicht irrational verknüpften Geraden durch O ist ebenfalls typisch. Das ∞^6-System $M_2^3(O^2)$ mit

$C_2(O)$ und zwei jetzt aber nothwendig irrational verknüpften Geraden durch O ist typisch. Das aus ihm durch Festlegung eines freien einfachen Basispunktes A_4 ausgeschiedene ∞^5-System ist nicht typisch, sondern unter die vorhin erhaltenen Typen zu bringen.

$M_2^4(O^3)$. C_7 sind die Bilder der Complexcurven, als Ergänzungscurve der C_4 erscheint eine C_3. Für $A_1^2 \equiv O_1$, $A_2^2 \equiv O_2$ muss $C_3 \equiv \Omega_3$ sein und es entstehen als Complement die Geraden OO_1, OO_2 zweimal gezählt. Das bedeutet, dass die anderen Monoide das gegebene längs jeder dieser zwei Geraden berühren sollen. Also erhalten wir als typisch entsprechend den $\infty^8 M_1^4(A_1^2 A_2^2)$ die $\infty^9 M_2^4(O^3)$ mit festem Osculationskegel 3. O. in O und Berührung längs jeder von zwei sich in O schneidenden Geraden.

Die darin enthaltenen Untersysteme können wir durch Verlegung der A_3 bis A_8 auf die Punkte O_i repräsentiren, wodurch die Ergänzungs-C_3 schon bei $A_1^2 A_2^2 A_3 A_4$ frei von der Identität mit Ω_3 wird. Aber gerade bei dieser freien Wahl entstehen für $\ldots\ldots A_3 A_4 \ldots\ldots A_3 A_4 A_5$ keine neuen Typen. Für $\ldots\ldots A_3 A_4 A_5 A_6$ nur dann, wenn $A_1 \ldots\ldots A_6$ nicht conconisch sind, aber für $A_3 \ldots\ldots A_7$ und $A_3 \ldots\ldots A_8$ bekommen wir unter dem Gesichtspunkte des rein Rationalen zwei neue Systeme. Also: Typen sind Systeme von $M_2^4(O^3)$ mit zwei Geraden durch O und einer diese nicht schneidenden Curve $C_6(O^3)$, $C_7(O^4)$, $C_8(O^5)$.

Lassen wir aber $C_3 \equiv \Omega_3$ sein, so geben $\ldots\ldots A_3 A_4 \ldots\ldots A_3 A_4 A_5$ zwei neue Typen und ebenso auch $\ldots\ldots A_3 A_4 A_5 A_6$, $\ldots\ldots A_3 A_4 A_5 A_6 A_7$, $\ldots\ldots A_3 A_4 A_5 A_6 A_7 B_8$, nämlich $M_2^4(O^3)$ mit festem Osculationskegel in O, Berührung längs zweier Geraden und ausserdem $2, \ldots, 6$ einfachen Geraden durch O in solcher irrationaler Verknüpfung, wie sie in der bezogenen Arbeit aus den Monatsheften beschrieben ist.

Alineationen von je 3 Punkten in diese Typen einzuführen, hat hier keine Schwierigkeit. Sollen aber 4 Punkte alineirt sein, so muss, wenn dieselben nicht frei in der Ebene sein sollen, das M_2^4-System also ein Untersystem eines der höheren sein soll, das durch Hinzunahme einfacher Raumpunkte entsteht, Ω_3 in eine Gerade und einen Kegelschnitt zerfallen. So entsteht ein typisches System von $M_2^4(O^3)$ mit festem Osculationskegel, Berührung längs zweier Geraden derselben, mit einer gemeinsamen Curve $C_4(O^3)$. Auch diese Quadrupelalineation kann man noch mit Tripelalineationen verknüpfen.

$M_2^5(O^4)$. Wir bedürfen dieser Monoide für die Situirung von C_4-Systemen, um den sämmtlichen Alineationen Rechnung zu tragen, welche unter den

$A_1, \ldots, A_8$ stattfinden können. Daher ist es auch passend, den letztvorigen Typus durch $M_2^5(O^4)$ zu ersetzen. Als Stammsystem können die $M_2^5(O_4)$ mit festem Osculationskegel und 5 einfachen Geraden durch O, längs denen sie sich berühren, angesehen werden.

$$C_5\text{-}Typus\,(A_1^2 \ldots A_5^2\, A_6 \ldots).$$

$M_2^2(O)$ *und* $M_2^3(O^2)$ gestatten ersichtlich nicht die Situirung dieses Typus in ihre Abbildungen.

$M_2^4(O^3)$ erfordert Ergänzungs-C_2 durch 7 Punkte auf Ω_3, also ist $C_5(A_1^2 \ldots A_5^2)$ nicht möglich. Dagegen kann das ∞^2-System $C_5(A_1^2 \ldots A_5^2 \overbrace{A_6 A_7 A_8})$ erreicht werden. C_2 ist dann das Bild einer $C_4(O^2)$. Also: das ∞^3-System von $M_2^4(O^3)$ mit 5 Geraden durch O und einer Curve $C_4(O^2)$ ist typisch, wenn die 5 Geraden irrational verknüpft sind.

$M_2^5(O^4)$ aber liefert weitere hieher gehörige Typen. Bilder der Complexcurven sind C_9 durch $O_1 \ldots O_{20}$, auf einer Curve Ω_4. Werden $A_i \equiv O_i$ gemacht, so erhält man als Ergänzungscurve eine Curve 4. O. durch $O_6 \ldots O_{20}$. Macht man $C_4 \equiv \Omega_4$, so entsteht ein typisches ∞^6-System von $M_2^5(O^4)$ mit festem Osculationskegel und Berührung längs 5 Geraden durch O. Ein anderes könnte entstehen, wenn durch die 15 O_i eine von Ω_4 verschiedene C_4 gienge, was aber nicht sein kann.

$M_2^6(O^5)$ erst liefert diesen gewünschten zweiten Typus. C_{11} sind die Bilder. Ergänzungscurve wird C_6 durch $O_6, \ldots, O_{30}$ und das System ist $M_2^6(O^5)$ mit Basiscurve $C_{11}(O^5)$ ferner mit 5 irrational verknüpften Geraden durch O, welche die C_{11} sonst nicht schneiden. Beim ∞^2-System C_5 mit $A_6 A_7 A_8$ ist die Alineation leicht anzubringen.

$$C_6\text{-}Typus\,(A_1^3 A_2^3 A_3^2 A_4^2 A_5^2 A_6 \ldots).$$

$M_2^5(O^4)$ liefert erst brauchbare Abbildungen. Die Ergänzungs-C_3 muss bei ∞^2-C_6 durch 11 O_i gehen. Es entstehen $\infty^2 M_2^5(O^4)$ mit fester $C_4(O)$, Berührung längs dreier Geraden OO_3, OO_4, OO_5 und Osculation längs OA_1, OA_2, dann ein ∞^4 mit fester C_3 (nicht durch O) und derselben Geraden wie vorhin; dagegen sind ∞^5 mit festem Osculationskegel hier nicht möglich.

$M_2^6(O^5)$. Die Ergänzungs-C_5 muss beim ∞^6-System durch 25 O_i gehen. Lassen wir sie mit Ω_5 coincidiren, was nothwendig ist, da sonst $(O_1 \ldots O_5) \equiv (A_1 \ldots A_5)$ in einer Geraden sein müssten, so entsteht ein typisches ∞^7-System $M_2^6(O^5)$ mit festem Osculationskegel mit 3 einfachen Geraden OO_3, OO_4 OO_5 und Berührung längs zwei Geraden OO_1, OO_2. Untersysteme haben

Geraden durch O gemeinsam. Alineationen sind leicht anzubringen. Für die Irreductibilität ist hier allemal die betreffend vorgeschriebene (l. c.) irrationale Verknüpfung unumgänglich.

§5.—*Die rational distincten Typen linearer Systeme elliptischer Curven auf unicursalen M_2.*

Da der Curvencomplex rational gegeben ist, also auch das erzeugende M_{r-1}-System, so ist auch das erzeugte M_2-System rational distinct gegeben. Man kann also durch Vorschreibung rational gegebener Punkte eine der erzeugten M_2 rational distinct auswählen. Auf der so gewonnenen rational distincten M_2, von der wir überdies nach § (cf. das dortige Citat) wissen, dass sie (wenn auch mit Zuhilfenahme *accessorischer* Irrationalitäten) unicursal ist, muss das rational distinct erzeugte ∞^d-System elliptischer Curven sich nicht so verhalten wie das rational distinct gegebene ∞^d-System in der Ebene. Dieses hängt davon ab, dass durch die Abbildung irrationale Gebilde in das Bildcurvensystem eintreten können. Dies ist so zu verstehen:

Zu einem rational distinct gegebenen linearen Curvensysteme L ∞^i, $i > 2$, gibt es stets eine rational distincte Fläche M_2, deren ebene Schnitte durch jene Curven (1, 1)-deutig abgebildet werden; desgleichen für ein M_{r-2}-System im Q_{r-1} eine M_{r-1} des R_r aber zu einer rational distinct (etwa durch ihre algebraische Gleichung) gegebenen Fläche gibt es nicht immer eine rational distincte Abbildung derselben. Ich will M_2 die zu L hinzuconstruirte Fläche nennen.* Die Construction geschieht, indem man in allgemeinster Weise ("allgemein" bezüglich auf Geschlecht, Ordnung und Singularitäten) ein ∞^3-System auswählt, seine Curven als Individuen den ∞^3 Ebenen eines R_3 und hiemit die ∞^2-Systeme mit einem gemeinsamen Punkte P' den ∞^2 Ebenenbündeln mit Scheiteln P, die die gesuchte M_2 erfüllen werden, entsprechen macht. Desgleichen, wenn man alle ∞^i-Curven den ∞^i R_{i-1} eines R_i zuweist. Aus der letzteren M_2 geht die erstere durch Projection aus $i-3$ Punkten hervor.

Was aus der rational distincten Abbildung als Folge rational distinct existirt, muss auch auf der hiezu construirten M_2 rational distinct vorhanden sein und was dort rational distinct abgeleitet wird, muss auch auf M_2 rational distinct abgeleitet werden können. So kann es kommen, dass die Abbildung

* Ohne Erwähnung des Rationalen hat dieses Verfahren des Hinzuconstruirens zuerst Caporali ("Sui sistemi lineari tripli" in den Coll. math., 1879, Rom) in ausgiebigem Maasse verwendet.

rational gegebene Elemente—Punktgruppen—implicirt, deren Existenz auf M_2 die Transversalcurven eines gewissen Curvensystemes C—in unserem Falle mit $p = 1$—ändert (vermehrt), und eine Folge davon wird sein, dass diese Transversalcurven, weil sie jene implicirten Gebilde auf der einzelnen M_2 in Auspruch nehmen, nicht zu Transversal-M_{r-1} des ganzen Complexes elliptischer Curven C Anlass geben. Hieher gehört nun das folgende wichtige Theorem:

THEOREM V.—*So oft es zu einem rational distinct gegebenen linearen Systeme von Curven (M_{r-1}) ein rational distinct daraus ableitbares Curven—(M_{r-1})—System gibt, kann man zwei Flächen (M_r), welche durch ein im ersten oder zweiten Systeme enthaltenes lineares System (1, 1)-deutig auf die Ebene (R_r) abbildbar sind, auch in rational distincter Weise eindeutig auf einander abbilden.*

Denn vermöge der (zu verfolgenden) Construction der beiden M_r wird jedem Punkte der einen ein Punkt des R_r und derselbe Punkt wird einem Punkte der anderen M_r entsprechen.*

Hier ist Zweck und Absicht, die Natur der ∞^d-Systeme von Curven $p = 1$ auf den rational distincten M_2 zu erforschen. Für $i \geq 3$ können wir in einer von den Riemann'schen Methoden nicht verschiedenen Weise die M_2 eindeutig distinct—wenn auch nicht durch birationale Raumtransformationen—auf eine andere Fläche Y_2 abbilden, deren ebene Schnitte $p = 1$ haben. Y_2 wird nöthigenfalls unter Einziehung von Irrationalitäten auf ein Abbildungssystem L von Curven $p = 1$ abbildbar sein. Die Irrationalitäten werden als rationale Gruppen zweiter oder höherer Art erscheinen. Wird L räumlich birational transformirt, so wird an der Construction der Fläche, und wird L besonders durch die in der bezogenen Arbeit definirten rational distincten Transformationen transformirt, auch nichts an den Rationalitätsverhältnissen auf Y_2 geändert. Die Typen der L-Systeme für $p = 1$ habe ich hier oben in §3 beschrieben. Gegenwärtig muss also

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* Mit Hilfe des Principes der "Verminderung der Functionen ϕ" hat F. Enriques in den Math. Ann., Bd. XLIX. diejenigen Classen unicursaler Flächen bestimmt, deren Individuen sich gegenseitig (1, 1)-deutig ohne Einführung numerischer Irrationalitäten auf einander abbilden lassen.

Aber gerade bei *dieser* Untersuchung ist das Princip der Verminderung nicht der wahre Grundgedanke und der geschickte italienische Verfasser hat nicht erkannt, dass das obige Theorem V seine Frage beherrscht, mit der ich mich übrigens hier gar nicht zu beschäftigen habe, da für die gegenwärtige Theorie und meine Behandlungsmethode derselben eine in jener Frage nur als kleines Stück erscheinende und nicht einmal in derselben Form zu stellende Frage erforderlich ist. Für jene Frage kann er sich nicht nur die Priorität zuschreiben, sondern ich will überhaupt an jener Frage als solcher einstweilen gar keinen Antheil haben. Also: die Verminderung der ϕ ist für seinen Gegenstand nur ein Mittel, um mein obiges Th. V in durchaus einheitlicher Weise anwenden zu können.

jeder derselben als Abbildungssystem einer nachzuconstruirenden Y_2 benützt werden. Diese Flächen werden alle rational von einander unterschieden sein, wäre es auch durch die Trennung der Geraden od. dgl. in rationale Gruppen. Der zweite Schritt wird sein, die auf diesen rational distincten Flächen Y_2 vorhandenen rational von einander verschiedenen linearen Systeme elliptischer Curven zu untersuchen.

Für den ersten Schritt wären die C_3, C_4, C_5, C_6-Typen der bezogenen Arbeit (Monatshefte, 1899) zu verwenden, aber da es uns hier nicht auf die Abzählung aller rational distincten Y_2, sondern auf die Präparirung für unseren zweiten Schritt ankommt, mögen wir die C_3, C_4-Typen, auf welche die C_4, C_5, C_6 *irrational* zurückgeführt werden können, allein verwenden, dann aus den gefundenen Y_2 die rational allgemeinsten Y_2 abstrahiren und an dieser Abstraction nun die Aenderungen bezüglich Irrationalität distinguiren.* Somit gestaltet sich der erste Schritt so:

$\infty^3 C_3$ durch 6 Punkte bilden eine F_3 mit rationalem Geradensextupel (das aber in niedere rationale Gruppen zerfallen kann) ab. Die Untersuchung der Abbildung lehrt 27 Geraden kennen und führt also durch Abstraction zu einer F_3 ohne rational distinctes Geradensextupel.

$\infty^3 C_3$ durch 5 Punkte bilden F_4 mit Doppelkegelschnitt und rationalem Geradenquintupel ab, führt also durch Abstraction zu allgemeinen F_4 mit Doppel-C_2. Ein Unterfall tritt ein, wenn drei Punkte alineirt sind. Die F_4 hat dann einen Doppelpunkt und ist rational distinct auf die Ebene abbildbar.

$\infty^3 C_3$ durch 4 Punkte bilden F_5 mit Doppelcurve 5. O. mit 3-fachem Punkte ab. Diese Y_2 lässt sich auf die F_3 abbilden gemäss dem Vorausgesandten. Im vorliegenden Falle ist das durch die $4A_i$ und die drei Nebenecken laufende C_3-Netz rational distinct ableitbar, also auch ein im Netze enthaltenes Büschel und das Paar weiterer Basispunkte, dann das $\infty^3 C_3$-System durch dieses Paar und die $4A_i$. Dieses ∞^3-System führt dann zur Abbildung auf eine F_3 und dann auf eine Ebene, weil ein rationales Geradenpaar der F_3 bekannt ist.

Zu einem anderen Beweise kann man sich aber des im ∞^3-Systeme enthaltenen Systemes in $C_1 + C_2$ zerfallender C_3 bedienen, deren C_1 sämmtlich einem einzigen Strahlbüschel angehören. Der Scheitel dieses Büschels ist ein rational

* Es kostet mich einige Überwindung, die Aufzählung der rational distincten M_2, wie sie sich vollständig aus meiner früher gegebenen Aufzählung der rational distincten linearen Systeme elliptischer Curven in der Ebene ergeben würde, wegzulassen.

distinct ableitbarer Punkt P_1. Sein conjugirter Punkt P_2 bezüglich $A_1 A_2 A_3 A_4$ ist ebenfalls rational distinct und $P_1 P_2$ bestimmen mit dem Vierecke wieder die Basis für das Abbildungssystem einer F_4 oder auch vermöge der beiden Strahlbüschel, deren Scheitel P_1, P_2 sind, sofort Abbildung auf die Ebene.

$\infty^3 C_3$ durch 3 Punkte bilden F_6 mit Doppelcurve 9. O. ab. Im Falle der abstrahirten Fläche fehlt die rationale Abbildbarkeit. Wir sahen im §1 dieser Arbeit, dass wir diesem merkwürdigen Systeme *kein rational distinctes Transversalensystem* zuordnen können.

$\infty^3 C_3$ durch 2, 1, 0 Punkte bilden F_7, F_8, F_9 ab. Dieselben können aber rational distinct auf die Ebene abgebildet werden. Denn (cf. §1) in allen drei Fällen ist das Geradensystem der Ebene ein *rational distinctes Transversalensystem* und ist auch rational distinct aus dem gegebenen C_3-Systeme ableitbar.

Es ist eben ein Corollar des Theoremes V, dass wenn aus dem Abbildungssysteme einer M_r das ∞^r-R_{r-1}-System des R_r rational distinct ableitbar ist, dass dann die $M_r(1, 1)$-deutig auf den R_r abbildbar ist.

$\infty^3 C_4$ durch $A_1^2 A_2^2$ gestatten die rational distincte Ableitung des Systemes der Kegelschnitte durch $A_1 A_2$; also ist nach meinem Theorem IV die M_2^8, die durch jene abgebildet wird, (1, 1)-deutig abbildbar auf die M_2^2, welche durch diese abgebildet wird.* In der That ist die M_2^2 *ohne rationalen Punkt* ein Typus.

Wir schliessen aus dieser Discussion:

THEOREM VI.—*Alle linearen $\infty^{4>2}$-Systeme elliptischer Curven im R_3 oder R_r, welche nur zwei Dimensionen erfüllen, sind von derselben Art, wie entweder* 1. *die Systeme in einer Ebene oder* 2. *auf einer M_2^2 oder* 3. *auf einer M_2^3 oder* 4. *auf einer M_2^4 mit eigentlichem oder zerfallendem Doppelkegelschnitt oder* 5. *auf einer M_2^6 mit Doppelcurve* 9. *Ord.*

Hiezu kommen die M_2^3 mit irrational verknüpften 2, 3, 4 Doppelpunkten, ebenso für No. 4. die $M_2^4(C_2^2)$ mit irrational verknüpften Doppelpunkten.

Um nun den zweiten Schritt auszuführen und dies eben in der oben genannten verkürzten Art, haben wir noch auf M_2^2, M_2^3, M_2^4, M_2^6 rational distinct Systeme von Curven $p = 1$ zu erzeugen, nachdem wir die in M_2^1, also der Ebene befindlichen bereits in §1 beschrieben oder kurz erwähnten.

Wir benützen hiezu die irrationale Abbildung—und dies ist der Kern unserer Methode, wie sie aus dem in §2 und dem vorher in diesem § Gesagten resultirt—

* Auch wird man aus dem Obigen schliessen, dass die M_2^4 mit Doppelgerade, wenn auf ihr ein lineares ∞^3-System elliptischer Curven rational distinct existirt, auf die Typen von oben abbildbar ist.

und disponiren in der Abbildung das Bild des auf der M_2 zu erzeugenden (als rational distinct gewünschten) Systemes. Hiebei sind aber nur jene Fälle beizubehalten, wodurch dieses Schneiden mit anderen M_2 die supponirte M_2 noch nicht ihren Typus des Theoremes VI verliert. Das soll den Schluss dieses § bilden und wir beginnen, da No. 1 schon in §4 zur I. Classe geführt hat, mit der der No. 2 entsprechenden II. Classe.

*Die zweifach ausgedehnten ∞^d-Systeme der II. Classe (M_2^2).**

Auf M_2^2 gibt es ein einziges System von Curven $p = 1$, bei dessen Ausschneiden M_2^2 nicht ihren Character des Theoremes VI verliert, dasjenige, dessen Abbildung die $C_4(O_1^2\, O_2^2)$ sind. Sie sind ausgeschnitten von allen M_2^2 des R_3.

Die zweifach ausgedehnten ∞^d-Systeme der III. Classe (M_2^3).

Dank den früher gegebenen rein rationalen Transformationen der ∞^d-Systeme elliptischer Curven auf ihre Typen können wir uns auf die Untersuchung derjenigen Bildsysteme in der Ebene beschränken, welche den unter Beachtung des Rationalen typischen Character haben.

Von den ∞^3 Ebenen des R_3 wird ein rational distinctes System von C_3 ausgeschnitten. *Das ∞^4-System* $C_3(A_1 \ldots A_5)$ kann zum Schnitte mit M_2^2 nur dann ergänzt werden, wenn $A_1 \ldots A_5$ gleichzeitig Basispunkte der M_2^3-Abbildung sind; Ergänzung ist $(A_1 \ldots A_5\, O_6^2)^3$, Bild eines Kegelschnittes. Also:

Ist auf M_2^3 ein Kegelschnitt rational distinct bekannt, so ist ein System von $M_1^4\, p = 1$ rational ausschneidbar mittelst der M_2^2 durch den Kegelschnitt.

Ergänzung zum Schnitte mit M_2^3 ist $(A_1^2 \ldots A_5^2\, O_6^3)^6$, Bild einer $M_1^5\, p = 2$. Also:

Ist auf M_2^3 eine Curve 5. Ord. $p = 2$ rational distinct bekannt, so ist ein System von $M_1^4\, p = 1$ rational ausschneidbar mittelst der M_2^3 durch M_1^5.

Ergänzung zum Schnitte mit M_2^4 ist $(A_1^3 \ldots A_5^3\, O_6^4)^9$, Bild einer $M_1^8\, p = 7$. Also: Ist auf M_2^4 eine Curve 8. Ord. $p = 7$ rational distinct bekannt, so ist ein System von $M_1^4\, p = 1$ rational ausschneidbar mittelst M_2^4 durch M_1^5. Dies kann fortgesetzt werden über M_2^5 hinaus zu jedem n, aber die Basiscurve wird schon von M_2^5 an bedingen, dass sie in einer einzigen M_2^3 liege, was, wie wir sehen werden, für uns unbrauchbar ist. Uebrigens bemerken wir wohl, dass es sich hiebei stets um ein und dasselbe Curvensystem $\infty^4 M_1^4$ auf M_2^3 handelt, dass dieses nur von verschiedenen Flächensystemen ausgeschnitten werden kann.

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* d bezeichnet die *unbestimmt* gelassene Dimension.

Das ∞^5-System C_3 mit $(A_1 A_2 A_3 A_4)$ unter den Fundamentalpunkten O_i wird zum Schnitt mit M_2^2 nicht zu ergänzen sein, zum Schnitte mit M_2^3 durch $(A_1^2 \ldots A_4^2 O_5^3 O_6^3)^6$, Bild einer $M_1^4 p = 0$. Also:

Ist auf M_2^3 eine $M_1^4 p = 0$ (2. Art) rational bekannt, so ist auf derselben ein lineares ∞^5-System $M_1^5 p = 1$ ausschneidbar mittelst M_2^3 durch M_1^4.

Wird ergänzt zum Schnitte mit M_2^4 durch $(A_1^3 \ldots A_4^3 O_5^4 O_6^4)^9$, Bild einer $M_1^7 p = 4$. Also: Ist auf M_2^3 eine $M_1^7 p = 4$ rational bekannt, so ist auf derselben ein lineares ∞^5-System $M_1^5 p = 1$ ausschneidbar mittelst M_2^4 durch M_1^7. Weiter müssen wir nicht gehen.

Das ∞^6-System C_3 mit $A_1 A_2 A_3$ unter den O_i wird zum Schnitte mit M_2^3 durch $(A_1^2 A_2^2 A_3^2 O_4^3 O_5^3 O_6^3)^6$, Bild dreier windschiefer Geraden, ergänzt. Hier müssen wir zum ersten Male beachten, dass nach §3 das Geradensystem als rational distinct für das C_3-System anzusehen ist, wenn ein Punkt A_i rational bekannt ist. Um also den Typus M_2^3 nicht zu verlieren und auf §4 zu fallen, müssen wir $A_1 A_2 A_3$ irrational verknüpft voraussetzen. In der That ist von den 3 windschiefen Geraden eine rational bekannt, so kann man die M_2^3 durch die Congruenz über die beiden anderen Geraden rational auf die Ebene abbilden und das System von M_2^3 in ein monoidales verwandeln. Also:

Ist auf M_2^3 ein irrational verknüpftes Tripel windschiefer Geraden bekannt, so ist auf derselben ein lineares ∞^6-System $M_1^6 p = 1$ ausschneidbar mittelst M_2^3 durch das Tripel.

Wird auch ergänzt durch $(A_1^3 A_2^3 A_3^3 O_4^4 O_5^4 O_6^4)^9$, Bild einer $M_1^6 p = 1$ zum Schnitte mit M_2^4. Durch die weiteren Curven ist nur eine einzige M_2^3 möglich und wir brechen treu unseren obigen Vorsätzen ab.

Das ∞^7-System C_3 mit $A_1 A_2$ unter den O_i wird zum Schnitte mit M_2^3 nicht zu ergänzen sein, zum Schnitte mit M_2^4 durch $(A_1^3 A_2^3 O_3^4 O_4^4 O_5^4 O_6^4)^9$, Bild eines Geradentripels; da aber zwei windschiefe Geraden darunter sind, so verliert die M_2^3 ihren typischen Character. Wird auch ergänzt durch $(A_1^4 A_2^4 O_3^5 \ldots O_6^5)^{12}$, Bild einer $M_1^8 p = 3$ zum Schnitt mit einer einzigen M_2^3.

Das ∞^8-System C_3 liefert durch $(A_1^4 O_2^5 \ldots O_6^5)^{12}$, Bild einer $M_1^6 + M_1^1$, die sich nicht schneiden, ebenfalls kein neues System und desgleichen das *∞^9-System C_3*.

Würden wir in einem dieser ∞^d-Systeme einen Punkt A_i ausserhalb der O_i angenommen haben, so hätten wir einfach ein Untersystem des jedesmal nächsten erhalten.

Das ∞^4-System C_4 mit $(A_1^2 A_2^2 A_3 A_4 A_5 A_6)$ unter den O_i wird zum Schnitte mit M_2^2 durch $(A_3 A_4 A_5 A_6)^2$, Bild eines Kegelschnittes, ergänzt, was oben vorkam, ebenso die weiteren Ergänzungen; das ∞^5-System zum Schnitt mit M_2^3 durch $(A_1 A_2 A_3^2 A_4^2 A_5^2 A_6^3)^5$, Bild von $M_1^4\, p = 0$, was oben vorkam, ebenso mit M_2^4, M_2^5, ...; das ∞^6-System zum Schnitt mit M_2^3 durch $(A_1 A_2 A_3^2 A_4^3 A_5^3 A_6^3)^5$, Bild dreier Geraden, die sich nicht schneiden; das ∞^8-System zum Schnitt mit M_2^4 durch $(A_1^2 A_2^2 O_3^4 \ldots O_6^4)^8$, Bild zweier windschiefer Geraden. Wir dürfen aber den C_4-Typus noch nicht verlassen. Werden $A_1^2 A_2^2$ ausserhalb der O_i genommen, und hiezu $A_3 \ldots A_6$ auf der O_i, so ergänzt $(A_1^2, \ldots A_6^2)^5$ zum Schnitte mit M_2^3, Bild einer $M_1^3\, p = 0$, deren Kenntnis die M_2^3 rational auf die Ebene abbildbar macht, desgleichen mit M_2^4, M_2^5 und werden die ∞^3-bis ∞^8-Systeme von C_4 genommen, so erhält man entweder nur eine M_2^3 oder Unmöglichkeit der Ergänzung. Wird nur A_2^2 ausserhalb der O_i genommen, so erhält man beim ∞^8-Systeme Ergänzung durch $(O_1^2 O_2^2 O_3^4 \ldots O_6^4)^8$, was nicht möglich ist, oder $(O_1^3 O_2^5 \ldots O_6^5)^{11}$, was zwei windschiefe Geraden implicirt; beim ∞^6- und ∞^5-Systeme desgleichen; das ∞^4-System $(O_1^2 O_2 O_3 O_4 O_5 A_2^2)^4$ mit A_2 ausserhalb der O gibt als Ergänzung zum Schnitt mit M_2^3 oder M_2^4 die Curve $(O_1 O_2^2 O_3^2 O_4^2 O_5^2 O_6^3)^5$, Bild von Gerade und Kegelschnitt, welche sich nicht schneiden. Sind also auf M_2^3 ein Punkt, eine Gerade M_1^1 und ein Kegelschnitt M_1^2 rational bekannt, wo M_1^1, M_1^2 sich nicht schneiden, so kann ein M_1^6-$(p = 1)$-System auf derselben ausgeschnitten werden mittelst M_2^3, welche M_1^1, M_1^2 enthalten und im Punkte die M_2^3 berühren. Das ∞^3-System C_4 wird durch $(O_1 O_2^2 \ldots O_6^2)^5$, Bild einer $M_1^4\, p = 1$ zum Schnitt mit M_2^3 ergänzt.

Also: sind auf M_2^3 ein Punkt und eine Curve $M_1^4\, p = 1$ rational bekannt, so kann ein ∞^3-System $M_1^5\, p = 1$ ausgeschnitten werden mittelst M_2^3, welche M_1^4 enthalten und die M_2^3 im Punkte berühren. Aber beide Systeme sind äquivalent Untersystemen früherer.

Die C_5- und C_6-Systeme liefern, insofern es möglich ist, sie zu ergänzen, nur dieselben räumlichen Curvensysteme wie bisher. In den meisten Fällen tritt die M_2^3 aus ihrem Typus heraus.

Die zweifach ausgedehnten ∞^d-Systeme III. Classe (M_2^4).

Die ebenen Schnitte bilden ein typisches ∞^3-System auf $M_2^4\,(C_2^2)$, aber dieses ist nicht vollständig. *Das ∞^4-System* von $M_2^2\,(C_2)$ schneidet M_2^4 in einem ∞^4-Systeme elliptischer Curven M_1^4. Dasselbe System wird auch ausgeschnitten von

M_2^3 durch C_2 und eine $M_1^4\ p=1$, welche C_2 vierpunktig schneidet und auch durch die $M_2^4(C_2^2)$ mit fester $M_1^4\ p=1$ und durch die $M_2^5(C_2^2)$ mit fester $M_1^8\ p=5$.

Das ∞^5*-System* von $C_3(A_1 \ldots A_4)$ erfordert als Ergänzung zum Schnitte mit $M_2^3(C_2)$ eine Curve $(A_1 \ldots A_4,\ O_5^2)^3$, Bild einer $M_1^4\ (p=0)$. Also:

Ist auf einer $M_2^4(C_2^2)$ ein windschiefe Curve 3. O. oder eine $M_1^7\ p=3$ bekannt, so kann ein ∞^5-System von $M_1^7\ p=1$ mittelst $M_2^3(C_2)$ oder $M_2^4(C_2^2)$ beide durch die M_1^3 oder mittelst $M_2^5(C_2^2)$ oder $M_2^6(C_2^3)$ durch $M_1^7\ p=3$ ausgeschnitten werden.

Das ∞^6*-System* von $C_3(A_1 A_2 A_3)$ erfordert als Ergänzung zum Schnitt mit $M_2^3(C_2)$ eine Curve $(O_4^2\ O_5^2\ O_1\ O_2\ O_3)^3$, Bild von zwei windschiefen Geraden, die irrational verknüpft sein müssen. Zum Schnitte mit $M_2^4(C_2^2)$ durch $(O_4^2 O_5^2 O_1 O_2 O_3)^3$ dasselbe wie soeben und mit $M_2^5(C_2^2)$ durch $(O_4^3\ O_5^3\ O_1^2\ O_2^2\ O_3^2)^6$, Bild einer $M_1^6\ p=1$ und desgleichen mit $M_2^6(C_2^3)$.

Das ∞^7*-System* $C_3(O_1,\ O_2)$ erfordert als Ergänzung zum Schnitte mit $M_2^5(C_2^2)$ eine Curve $(O_3^3\ O_4^3\ O_5^3\ O_1^2\ O_2^2)^6$, Bild einer Geraden und einer $M_1^4\ p=0$ 2. Art, die sich nicht schneiden. Zum Schnitte mit M_2^3, die nicht durch C_2 geht, desgleichen. Das System dieser $M_2^4(C_2^2)$ ist reductibel auf ein System M_2^3, weil die Gerade rational gegeben sein muss.

Das ∞^8*-System* $C_3(O_1)$ wird ergänzt zum Schnitt mit M_2^3 durch $(A_1^3 A_2^3 \ldots A_5^3)^6$, also jedenfalls eine rat. dist. Gerade, was die M_2^4 auf M_2^3 reductibel macht, zum Schnitte mit M_2^4 durch $(O_1^4\ O_2^4\ O_3^4\ O_4^4\ O_5^3)^9$, Bild von M_1^1 und $M_1^7\ p=2$, mit $M_2^4\ (C_2)$ durch $(O_1^3\ O_2^3\ O_3^3\ O_4^3\ O_5^2)^6$, Bild von M_1^1 und M_1^3, mit $M_2^4(C_2^2)$ durch $(O_1^2\ O_2^2\ O_3^2\ O_4^2\ O_5)$, was unmöglich ist.

Das ∞^9*-System* C_3 wird durch $(O_1\ O_2\ O_3^3\ O_4^3\ O_5^3)^5$ zum Schnitt mit M_2^5 ergänzt; dies sind 4 windschiefe Geraden, aber durch sie geht nur eine einzige $M_2^4(C_2^2)$.

Das ∞^8*-System* $C_4(O_1^2\ O_2^2)$ wird durch Bildcurven ergänzt, denen keine verwendbare Curve auf der $M_2^4(C_2^2)$ entsprechen kann. *Das* ∞^6*-System* $C_4(O_1^2 O_2^2 O_3 O_4)$ wird zum Schnitte mit M_2^2 ergänzt durch $(O_3\ O_4\ O_5^2)^2$, also in R_3 ein Paar windschiefer Geraden. Also: Ist auf einer $M_2^4(C_2^2)$ ein Paar windschiefer Geraden bekannt, so kann ein ∞^8-System von Curven $p=1$ mittelst M_2^2 oder $M_2^3(C_2)$ durch die Geraden ausgeschnitten werden.

Zum Schnitte mit M_2^3 durch $(O_1\ O_2\ O_3^2\ O_4^2\ O_5^3)^5$, Bild einer $M_1^6\ p=1$, zum Schnitt mit $M_2^3(C_2)$ durch $(O_3\ O_4\ O_5^2)^2$, mit M_2^4 durch $(O_1^2\ O_2^2\ O_3^3\ O_4^3\ O_5^4)^8$, Bild einer $M_1^{10}\ p=7$, durch welche nur eine $M_2^4(C_2^2)$ geht. Also bleibt nur $M_2^4(C_2)$ durch $M_1^6\ p=1$. Das ∞^5-System C_4 liefert nichts Neues.

Das ∞^5-C_5-*System* liefert $M_2^4(C_2^2)$ durch $M_1^3(p=0)$; eine Disposition einzelner Doppelpunkte ausserhalb der O^i ist unmöglich wegen der irrationalen Verknüpfung von $O_1^2 \ldots O_5^2$.

Das ∞^6-C_6-*System* liefert $(A_1^3 A_2^3 A_3^3 A_4^3 A_5^2)^6$ und wird zum Schnitt mit $M_2^4(C_2^2)$ durch zwei windschiefe Geraden ergänzt.

Die zweifach ausgedehnten ∞^d-*Systeme V. Classe* (M_2^6).

Abbildung der ebenen Schnitte geschieht durch $\infty^3 C_3(O_1 O_2 O_3)$. Das Bild der Doppelcurve 9. Ordnung ist in der Ebene eine Curve 9. O. durch $O_1^3 O_2^3 O_3^3$. Die ∞^6 Curven $C_3(O_1 O_2 O_3)$ sind Bilder der Schnitte von $M_2^6(C_9^2)$ mit M_2^4, welche die C_9 enthalten. Die C_9 ist also in einer M_2^3 enthalten und ist also die Basis eines ∞^1-Systemes von $M_2^6(C_9^2)$. C_9 hat $p=9$. Wir können daher auch keine M_2^5, ... zum Ausschneiden der Curvensysteme verwenden, da die Basiscurve die $M_2^6(C_9^2)$ vollständig bestimmen würde.

§6.—*Die eigentlichen Complexe elliptischer Curven im* R_3, *deren zweifach ausgedehnte Systeme von den Classen II bis V sind.*

1. In §4 sind die Complexe der I. Classe erschöpft worden. Nachdem wir nun die Typen der zweifach ausgedehnten Systeme elliptischer Curven, die in einem Complexe überhaupt noch vorkommen können, festgesetzt haben, müssen wir die Möglichkeit untersuchen, jene Complexe, in denen diese Typen als Untersysteme vorkommen, auf eine möglichst geringe Anzahl nicht äquivalenter Typen zu reduciren.

2. *Classe II.* Wie die Abbildung zeigt, gibt es für das einzige hiebei erhaltene typische System, nämlich ∞^8 M_1^4, ein rational distinctes Transversalensystem ∞^3, die Transversalen sind selbst rational. Also muss es für das ∞^9-System von M_2^n, wenn diese M_2^n den M_2^2-Typus haben, ein ∞^3-System von M_2^m geben, welche die M_2^n in rationalen Curven schneiden, weil diese Curven die M_2^n in Punktepaaren schneiden müssen, indem das Transversalensystem den Rang 2 hat. Das System ist ausserdem vollständig; daher folgt nach dem Lemma in Amer. Journ., XXIII, p. 13, dass es homoloidal im R_3 ist. Wird dieses also zu einer Transformation benützt, so erhalten wir die M_2^n verwandelt in M_2^2. Also:

Diejenigen M_2^n-*Systeme mit gegenseitigen elliptischen Schnitten, deren zweifach ausgedehnte Curvensysteme* (*also deren* M_2^n)—*vom* M_2^2-*Typus sind, sind raümlich birational äquivalent mit dem* ∞^9-M_2^2-*System oder einem durch* 2, 3, ... 8 *Punkte*

daraus ausgeschiedenen Untersysteme, wobei aber diese Punktgruppen so irrational verknüpft sein müssen, dass kein einzelner Punkt rational gegeben ist.

Denn, wenn ein Punkt rational gegeben wird, fällt das System unter den §4.

3. *Classe III.* Hier ist folgende Feststellung nothwendig: Geht man die typischen Systeme der Reihe nach durch, so sieht man, wie in Anbetracht der Transversalzahlen (cf. §1) das System der ∞^3 Ebenen des R_3 eine einzigartige Rolle inne hat. Es gibt kein anderes vollständiges lineares ∞^3-System von M_2, die die Curven des Complexes und die mit ihnen in Zusammenhang stehenden Curven in ebenso vielen Punkten schneiden würden wie die Ebenen also u. A. die Complexcurven resp. in 4, 5, 6, 6, 5 Punkten. Das Gesammtsystem der vorliegenden M_2^3 besitzt in sich dieselbe Transversalzahl und dieses System ist als M_2-System selbst rational distinct. Aber in diesem ist kein anderes System vom Range 1 enthalten, das die M_2^3, welche den Complex erzeugen, in elliptischen Curven mit gleicher Transverzahl schnitte.

Das *System der ebenen Schnittcurven* ist also für jene gefundenen Curvensysteme auf den M_2^3, also auch auf den M_2^n ihres Typus das entsprechende ∞^3-System, ein rational distinctes Transversalensystem;* wir schliessen mit Hilfe des in §2. u. 1. Gesagten, dass es dann auch ein System von M_2^m geben muss, welches die sämmtlichen Curven des Complexes in ebensovielen Punkten schneidet und also die M_2^n gerade *in jenen* ∞^3 *Transversalcurven* des rational distinct ableitbaren Systemes *schneidet.*

Diese M_2^m müssen nun ein System von Curven erzeugen (als gegenseitige Schnitte), die jene M_2^n in Punktetripeln treffen. Da wir nun aber mindestens ∞^3-Systeme M_2^n untersuchen und die ∞^2-Systeme noch nicht einer systematischen Discussion unterziehen wollen, können wir auf einer solchen Curve 2 Punkte nehmen, durch sie die M_2^n legen, welche noch ∞^1 sein werden und aber jede dieser Curven noch in einem variabeln Punkte schneiden, also sind diese Curven rational. Sonach ist aus jedem Curvencomplexe $p = 1$, dessen M_2^m vom M_2^3-Typus sind, auf rational distincte Art ein vollständiges lineares ∞^3-System von M_2 mit rationalen gegenseitigen Schnittcurven ableitbar und die Schnittcurven treffen die M_2^n in Punktetripeln. Nach dem mehrfach von mir angewandten Lemma† ist dieses System homaloidal. Wird es zu einer Raumtransformation

* Man würde den Sinn der Transversalenmethode misverstehen, wenn man daraus schliessen wollte dass auch das System der R_2 selbst für die Curven rationaldistinct sei.

† Amer. Journ. of Math., XXIII, p. 13.

verwendet, so wird durch diese M_2^n-Systeme eben in das M_2^3-System übertragen. Daher:

Diejenigen eigentlichen Complexe von Curven $p=1$, deren zweifach ausgedehnte Curvensysteme also (erzeugende M_2) von der Classe III. (M_2^3-Typus) sind, sind räumlich birational äquivalent mit

1. *dem ∞^5-System von M_2^3 durch $M_1^5\, p=2$,*
2. *dem ∞^6-System von M_2^3 durch $M_1^4\, p=0$ (2. Art),*
3. *dem ∞^6-System von M_2^3 durch ein irrational verknüpftes Tripel windschiefer Geraden,*
4. *einem der Systeme M_2^3 mit 2, 3, 4 irrational verknüpften Doppelpunkten und gegenseitigen elliptischen Schnittcurven,*
5. *einer der Degenerationen, die aus n. 1. durch Zerfallen von M_1^5 in a) $M_1^4\, p=1$ mit Bisecante oder b) zwei irrational verknüpfte M_1^2 und eine beide schneidende Gerade hervorgehen (die übrigen Degenerationen gehören nach Classe I.),*
6. *der Degeneration, die aus n. 2. durch Zerfallen von M_1^4 in zwei sich einpunktig schneidende Kegelschnitte entsteht,*

oder mit niederdimensionalen Systemen, welche aus diesen durch Annahme fester einfacher Basispunkte im R_3 entstehen.

Classe IV. Dieselbe Beweisführung wie für Classen II. und III. gilt für den $M_2^4\,(C_2^2)$-Typus und ergibt:

Jene eigentlichen Complexe von Curven $p=1$, deren zweifach ausgedehnte Curvensysteme also (erzeugende M_2) den M_2^4-Typus haben, sind sämmtlich birational äquivalent mit

1. *dem ∞^5-System von $M_2^4\,(C_2^2)$ durch $M_1^4\, p=1$,*
2. *dem ∞^6-System von $M_2^4\,(C_2^2)$ durch $M_1^3\, p=0$,*
3. *dem ∞^7-System von $M_2^4\,(C_2^2)$ durch zwei irrational verknüpfte windschiefe Geraden,*
4. *denselben drei Systemen, aber mit irrational zerfallendem Doppelkegelschnitt oder einem Systeme, das aus diesem durch Festsetzung irrational verknüpfter einfacher Basispunkte entsteht.*

§7.—*Die vollständigen ∞^3-Systeme mit elliptischen Complexcurven im R_3.*

Aus den Untersuchungen der §§4 und 5 können wir ein wichtiges Resultat ziehen: *Es gibt in R_3 kein vollständiges ∞^2-System einer der IV. Classen. Sie können alle aus höher dimensionalen Systemen durch Festsetzung einfacher Basispunktgruppen ausgeschaltet werden.*

Zur vollständigen Erkenntnis bleibt uns noch übrig, jene Flächenabbildungen zu discutiren, wo man von einem ∞^2-System $C_n\, p = 1$ auch nur zu einem ∞^2-Systeme von Curven geführt wird, das auf M_2 rational distinct vorhanden ist. Hiezu wird uns in Anwendung des Theoremes IV *jede* Fläche dienen können, die in die Ebene auf ein Curvensystem abgebildet wird, das aus dem ∞^2-System rational distinct ableitbar ist. Es gibt hier deren verschiedene. Die *inversen* Adjungirten* führen in einfachster Weise zum Ziele. Nämlich aus dem ∞^2-Systeme von C_3, C_4, C_5, C_6 mit resp. $(A_1, \ldots . A_7)$, $(A_1^2 A_2^2 A_3 \ldots . A_6)$, $(A_1^2, \ldots . A_5^2)$, $(A_1^3 A_2^3 A_3^2 A_4^2 A_5^2 A_6 \ldots . A_9)$ ist das System C_6, C_7, C_8, C_9 mit resp. $(A^2, \ldots . A_7^2)$, $(A_1^3 A_2^3 A_3^2 \ldots . A_6^2)$, $(A_1^3 \ldots . A_5^3)$, $(A_1^4 A_2^4 A_3^3 A_4^3 A_5^3 A_6^2 \ldots . A_9^2)$ rational distinct ableitbar. Wenn somit auf einer M_2 ein vollständiges ∞^2-System von Curven $p = 1$ erscheint (also ohne einfache Basispunkte) dessen Typus der der C_3, C_4, C_5 in der Ebene bei irrationaler Abbildung wird, so ist diese Fläche rational distinct auf eine M_2^8 abbildbar, deren Abbildungssystem aus den genannten C_6, C_7, C_8, C_9 besteht.

∞^6-System C_6. Wir erhalten eine Vereinfachung dadurch, dass mit den $\infty^2 C_3$ auch die einzelnen Büschel, also die einzelnen Punktepaare der Ebene, welche zu Basisnonupeln ergänzen, rational distinct gegeben sind. Setzen wir zwei solche Paare zu den 7 Doppelpunkten hinzu, so erhalten wir immer noch ein rational distinctes ∞^3-System. Durch diese 11 Punkte geht stets eine C_3; die Fläche, auf welche wir jetzt rational distinct abgebildet haben, ist die von Nöther† entdeckte M_2^4 mit Selbstberührungspunkt O; das Curvennetz ist das der C_4 mit Selbstberührungspunkt, welche von den Ebenen durch O ausgeschnitten werden.‡

Der letzte Typus von Flächen, auf denen ein ∞^2-System elliptischer Curven rational distinct ausgeschnitten werden kann, ist der, dessen Repräsentant die M_2^4 mit Selbstberührungspunkt ist.

Die ∞^5-Systeme C_7, C_8 und das ∞^4-System C_9 von hier vorher führen zu keinem neuen Typus. Für C_7 und C_8 ist das ∞^3-System C_2 durch A_1, A_2 rational distinct ableitbar, daher nach Theorem V. die Flächen M_2^7 und M_2^6 auf die M_2^2 und

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* Act. Math., Bd. XIX. "Neue Theorie etc."

† Gött. Nachr. 1870.

‡ In der von mir schon genannten Arbeit des F. Enriques in Math. Ann., Bd. XLIX, ist gesagt, dass die Flächen mit vollständigem ∞^2 Systeme elliptischer Curven die (1, 1)-deutige Abbildung auf eine elliptische Doppelebene gestatten. Damit ist aber nichts gewonnen und diese Angabe ist für die Anwendung werthlos. Hier ist der Ersatz durch die M_2^4 mit tacnode unerlässlich. Enriques kannte diesen Typus nicht.

wegen des beim C_9-System rational distincten Systemes $C_3(A_1 \ldots A_5)$ diese M_2^7 auf die $M_2^4(C_2^2)$ rational distinct abbildbar sind. Aehnliches ist für C_9 durchführbar.

Um nun die M_2^n-Systeme, deren M_2 diesem Typus angehören, auf Typen zu bringen, wenden wir wieder das Transversalenprincip aus §7 an. Zwar ist für die C_4 durch den Selbstberührungspunkt das Ebenensystem des R_3 nicht rational distinct, aber das M_2^2-System mit Berührung in O an den Flächen M_2^4. Die Schnittcurven mit M_2^4 bilden Transversalen, deren Analoge auch auf den M_2^n irgend eines Systemes unserer Classe wieder erscheinen müssen, also auch das ∞^6-System von Transversal-M_2^m, deren Schnittcurven C die M_2^n in Punktequadrupeln treffen und welches unter diesen Curven C ein ∞^2-Bündel rationaler Curven hat, die die M_2^n in Punktepaaren treffen. Wegen des ∞^3-Systemes M_2^n können wir auf allen Curven $C\infty^1$ quadratische Involutionen ausschneiden, indem wir je zwei Punkte für die M_2^n fest legen, also sind die C jedenfalls elliptisch. Ohne erst nachzuweisen, dass sie sogar rational sind, was sie thatsächlich sind, könnten wir im Falle $p = 1$ durch estlegung Feiner festen Berührungsebene der M_2^m ein homaloidales System derselben ausschalten, mittelst dem transformirt das M_2^n-System sich in ein ∞^3-M_2^4-System verwandeln muss, das das ∞^2-System von elliptischen Curven liefert. Eine weitere Discussion führt dann dazu, dass diese M_1^3 nur vom gegenwärtigen Typus sein können:

Jeder eigentliche Complex von Curven $p = 1$ in einem vollständigen ∞^3-M_2-Systeme, dessen M_2 vom Typus der M_2^4 mit tacnode sind, ist räumlich birational äquivalent einem ∞^3-Systeme von M_2^4 mit tacnode selbst, welche sich in einer Schnittcurve auf $M_2^3(O)$ schneiden.

§8.—*Die uneigentlichen Complexe elliptischer Curven im R_3, ihre Auffindung.*

Von den beiden erzeugenden M_2-Systemen sei eines vom M_2^1 oder M_2^2- oder M_2^3- oder M_2^4-Typus.

Classe I. In diesem Falle entsteht sofort ein ∞^2-System einpunktiger Transversalen und also das monoidale System. Diese Complexe sind in §3 untersucht worden bis auf jene, wo nur ein Büschel solcher Flächen vorhanden ist. Dies lässt sich stets rational distinct auf ein Ebenenbüschel beziehen. In den Ebenen können dann entweder Systeme vom C_3-, C_4-, C_5- oder C_6-Typus erscheinen, wodurch 4 Classen uneigentlicher Complexe entstehen.

Classe II. Das mögliche System der $\infty^6 M_1^4$ hat auf M_2^2 ein rational distinctes Transversalensystem, nämlich das der ∞^3 Kegelschnitte vom Range 2.

Solches muss auch auf jeder M_2^n eines der fraglichen Systeme sein, ausgeschnitten nach §2 von einem ∞^3 M_2^m-Systeme, von dem wie in §6 die Rationalität der gegenseitigen Schnittcurven und der Rang 1 bewiesen wird. Transformation damit führt auf ein M_2^2-System und wir finden: Uneigentliche Complexe mit einem erzeugenden Systeme vom M_2^2-Typus sind räumlich birational äquivalent einem Complexe, der von den M_2^2 und M_2^3 durch einen festen Kegelschnitt oder einen, der von den M_2^2 und M_2^4 durch eine feste $M_1^4\, p = 1$ erzeugt wird.

Classe III. Wir haben die Discussion der M_2^3 in §6 durchzugehen, zu bemerken, dass für die dort auf M_2^3 erzeugten Curvensysteme, welche jedesmal dieselben sind wie die in §7 verwendeten, die von den Ebenen ausgeschnittenen Transversal-C_3 rational distincte Systeme bilden, dass deren also auch auf den M_2^n desselben Typus erscheinen müssen, die dann wieder von einem Systeme M_2^m, das den Rang 1 erkennen lässt, ausgeschnitten werden, um mittelst einer Transformation zu schliessen: Uneigentliche Complexe, von deren erzeugenden Systemen keines der Classe I oder II ist, eines aber der Classe III ist, sind räumlich birational äquivalent einem Complexe erzeugt von M_2^3 und M_2^4 durch eine $M_1^8\, p=7$ oder $M_1^7\, p = 4$ oder $M_1^6\, p = 1$ oder durch $M_1^6\, p = 2$ nebst Berührung in einem festen Punkte oder durch $M_1^7\, p = 5$ nebst Berührung in einem freien Punkte. Hiezu kommt der uneigentliche Complex aller C_3, in denen die Ebenen des R_3 alle M_2^3 schneiden. Endlich können die Basiscurven M_1^6, M_1^7, M_1^8 sich zerlegen. Dies darf jedoch nur so geschehen, dass der M_2^3-Typus sich nicht ändert.

Classe IV. Hier müssen wir beachten, dass in einzelnen Fällen zwar nicht das System der ∞^3 Ebenen, wohl aber das der $\infty^2 M_2^2(C_2)$ ein rational distinctes Transversalensystem liefert. Die anzuwendenden Transformationen sind dieselben wie in §7; die nöthigen Vorarbeiten sind in §6 durchgeführt. Uneigentliche Complexe, von deren erzeugenden M_2-Systemen eines der Classe IV ist, sind räumlich birational äquivalent einem Complexe, erzeugt von den $M_2^4(C_2^2)$ und den M_2^3 durch C_2 und eine $M_1^3\, p = 0$ (ein System ist dann vom monoidalen T.) den $M_2^4(C_2^2)$ und $M_2^3(C_2)$ durch zwei windschiefe Geraden (desgleichen), den $M_2^4(C_2^2)$ mit den $M_2^4(C_2)$ durch $M_1^8\, p = 5$ oder $M_1^7\, p = 3$ oder $M_1^6\, p = 1$. Kommt es nur auf den Complex an, so ist der erste dieser Aufzählung identisch mit dem ersten der unter Classe III. hier aufgezählten,

Classe V. Die $M_2^6(C_3^2)$ geben zu einem Complexe elliptischer Curven Anlass vermöge der Ebenen des Raumes und dann vermöge der M_2^4 durch die Doppelcurve, hierüber §5.

$M_2^4(O^2)$ mit tacnode führt nur zu dem Systeme der Schnitte der sämmtlichen M_2^4 dieser Art mit gegebenem tacnode und gegebener Berührungsebene mit allen Ebenen durch diesen Punkt und zu keinem anderen Complexe, weil diese Curven nur mittelst Flächen ausgeschnitten werden können, durch deren Basiscurve höchstens eine solche M_2^4 hindurchgehen kann.

§9.—*Die Typen uneigentlicher und eigentlicher Complexe elliptischer Curven im R_3.*

Die gewonnenen Resultate sollen nun gesichtet und zusammengefasst werden. So entsteht das

THEOREM VII.—A. Die *uneigentlichen* Complexe elliptischer Curven sind auf rational distincte Art räumlich birational äquivalent den folgenden Typen.*

Classe I. Schnitt eines linearen ∞^u-Systemes elliptischer Kegeln mit gemeinsamem Scheitel O und von einem der früher gegebenen Typen mit einem linearen ∞^{n_1}-Systeme von Monoiden $M_2^n(O^{n-1})$. Hier ist n_1 unbeschränkt gross.†

Classe II. 1. Schnitt der M_2^2 mit den M_2^3 durch einen beiden gemeinsamen Kegelschnitt. Von den M_2^3 genügt auch ein System mit 9 festen Basispunkten ausserhalb den M_2^2.

2. Schnitt der M_2^2 mit den M_2^4 durch eine beiden gemeinsame $M_1^4\, p=1$, nebst Degenerationen durch Zerfallen der Basiscurve bei irrationaler Verknüpfung.

Classe III. 1. Schnitt der M_2^3 mit den M_2^4 durch eine beiden gemeinsame $M_1^8\, p=7$.

2. oder $M_1^7\, p=4$,

3. oder $M_1^6\, p=1$,

4. oder $M_1^6\, p=2$ und mit Contact in einem freien Punkte,

5. oder $M_1^7\, p=5$ mit Contact in einem freien Punkte.

Classe IV. 1. Schnitt der $M_2^4(C_2^2)$ mit den $M_2^3(C_2)$ durch eine beiden gemeinsame $M_1^4\, p=1$.

2. Schnitt der $M_2^4(C_2^2)$ und $M_2^4(C_2)$ durch $M_1^8\, p=5$,

3. der $M_2^4(C_2^2)$ und $M_2^4(C_2)$ durch $M_1^7\, p=3$,

4. der $M_2^4(C_2^2)$ und $M_2^4(C_2)$ durch $M_1^6\, p=1$,

5. der $M_2^4(C_2^2)$ und $M_2^3(C_2)$ wie n. 1 aber mit zwei irrationalen M_1^2,

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* Hier kommt es nur darauf an, die typischen Complexe zu finden. Anders gestaltet sich das Resultat bei der Frage nach den typischen Paaren von M_2-Systemen. Diese Frage kommt dann in der Theorie der Transformationsgruppen in die erste Reihe.

† Hier herein gehört die Gesammtheit aller ebenen C_3, $p=1$, etc.

6. Degenerationen von n. 3. bis n. 5 durch Zerfallen der M_1^8, M_1^7, M_1^6,

7. Degenerationen von n. 1 bis n. 6 durch Zerfallen der C_2 in zwei irrationale Gerade.

Classe V. 1. Schnitt von $\infty^1 M_2^6(C_9^3)$ mit den $M_2^4(C_9)$,

2. Degenerationen von n. 1. durch Zerfallen der C_9.

Hiezu kommen als Typen noch Untersysteme, die aus den aufgezählten Systemen durch Festsetzung einfacher Basispunkte entstehen, die entweder beiden erzeugenden M_2-Systemen oder nur einem derselben gemeinsam sind, und welche in einzelnen Fällen nicht rational sein dürfen.

B. Die *eigentlichen* Complexe sind auf rational distincte Art räumlich birational äquivalent folgenden Typen:

Classe I. 1. $\infty^u M_2^2$ durch einen rational gegebenen Punkt O und weitere irrational verknüpfte Punktgruppen, unter denen auch mit O complane Tripel sein können, wodurch eine typische Untereintheilung entsteht; $u < 9$.

C_3-*Typus*

2. $\infty^{10-\alpha} M_2^3(O^2)$ mit festem Osculationskegel in O und $\alpha = 0, 1, \ldots 6$ festen Geraden durch O oder im Falle $\alpha = 6$ mit 6 freien einfachen Basispunkten, welche in keinem Quadrikegel mit dem Scheitel in O enthalten sind.* Die Geraden oder Punkte können verschiedentlich irrational verknüpft sein,

3. $\infty^7 M_2^3(O^2)$ mit $M_1^3(O)$,

4. $\infty^6 M_2^3(O^2)$ mit $M_1^4(O^2)$,

5. $\infty^5 M_2^3(O^2)$ mit $M_1^5(O^3)$,

6. $\infty^4 M_2^3(O^2)$ mit $M_1^6(O^4)$,

7. $\infty^7 M_2^3(O^2)$ mit einer festen Osculationsebene in O und einer festen, nicht in dieser enthaltenen $M_1^3(O^2)$, die nicht zerfallen soll.

8. $\infty^4 M_2^3(O^2)$ durch zwei feste ebene Curven 3. O., welche in O eine gemeinsame Spitzentangente haben,

9. $\infty^4 M_2^4(O^3)$ mit fester $C_{10}(O^6)$,

10. $\infty^3 M_2^4(O^3)$ mit fester $C_{11}(O^9)$,

(Hieher würde auch das ∞^2 System $M_2^4(O^3)$ mit fester $C_{12}(O^{10})$ gehören).

* Wenn man nicht darauf hält, die Ordnungen der Complexcurven zu vereinfachen, kann man auch statt der Typen, die $\alpha = 1, \ldots 5$ Geraden durch 0 enthalten, solche wählen, wo $\alpha = 1, \ldots 5$ freie einfache Basispunkte vorhanden sind.

C_4-*Typus*

11. $\infty^7 M_2^3(O^2)$ mit zwei Geraden durch O und einer Geraden fremd zu O,

12. $\infty^5 M_2^3(O^2)$ mit zwei Geraden durch O und $M_1^3(O^2)$,

13. $\infty^9 M_2^4(O^3)$ mit festem Osculationskegel und Berührung längs zweier Geraden durch O,*

14. $\infty^7 M_2^4(O^3)$ mit zwei Geraden durch O und $M_1^4(O)$,

15. $\infty^6 M_2^4(O^3)$ mit zwei irrational verknüpften Geraden durch O und $M_1^5(O^2)$,

16. $\infty^5 M_2^4(O^3)$ mit zwei Geraden durch O und $M_1^6(O^3)$,

17. $\infty^4 M_2^4(O^3)$ mit zwei Geraden durch O und $M_1^7(O^4)$,

18. $\infty^3 M_2^4(O^3)$ mit zwei Geraden durch O und $M_1^8(O^5)$.

C_5-*Typus*

19. $\infty^6 M_2^4(O^3)$ mit 5 Geraden durch O, die irrational verknüpft sein müssen und $M_1^4(O^2)$,

20. $\infty^6 M_2^5(O^4)$ mit festem Osculationskegel und Berührung längs 5 irrational verknüpfter Geraden,

21. $\infty^6 M_2^5(O^9)$ mit $M_1^{11}(O^5)$ und einem rationalen Geradenquintupel durch O.

C_6-*Typus*

22. $\infty^6 M_2^5(O^4)$ mit $M_1^4(O)$ und Berührung längs dreier, Osculation längs zweier je irrational verknüpfter Geraden durch O,

23. $\infty^5 M_2^5(O^4)$ mit $M_1^3(O)$ und Geraden wie in 22.

24. $\infty^7 M_2^6(O^5)$ mit Geraden wie in 22.

Classe II. $\infty^{11} M_2^2$ mit einfachen Basispunktgruppen, unter denen kein einziger rational distincter Punkt sein darf. Untertypen entstehen durch complane Lagen von Quadrupeln dieser Punkte oder andererseits Berührungen, die aber theilweise jenen complanen Quadrupeln äquivalent sind.

Classe III. 1. $\infty^5 M_2^3$ durch $M_1^5\, p = 2$,

2. $\infty^6 M_2^3$ durch $M_1^4\, p = 0$, 2. Art.,

3. $\infty^6 M_2^3$ mit irrational verknüpftem Tripel windschiefer Geraden,†

4. $\infty^4 M_2^3$ mit $M_1^4\, p = 1$ mit Berührung in freiem Punkte,

5. M_2^3-Systeme mit 2, 3, 4 irrational verknüpften Doppelpunkten.

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* Enriques hat $M_2^4(O^3)$ mit 2 Doppelgeraden, was, so viel ich sehe, ebenfalls richtig ist.

† Diese Bemerkung ist ganz wesentlich. Denn sind die Geraden nicht irrational verknüpft, so sind die M_2^3 vom monoidalen Typus, weil sie mittelst einer rationalen Strahlencongruenz auf die Ebene abgebildet werden können. In der That kann man dieses System stets in ein monoidales verwandeln, also von der I. Classe.

6. Wie n. 1. aber mit Zerfallen von M_1^5 in $M_1^4\,(p=1)$ mit Bisecante.*

7. oder in zwei irrational verknüpfte M_1^2 und eine gemeinsame Transversalgerade,

8. oder in eine ebene M_1^3 und einen sie zweifach schneidenden Kegelschnitt, aber mit der ausdrücklichen Voraussetzung, dass man auf keiner dieser beiden Curven einen rational distincten Punkt kenne;† die Curven können irrational zerfallen.

9. Wie n. 2, aber mit Zerfallen von M_1^4 in zwei sich einpunktig schneidende M_1^2.

10. M_2^3 aus n. 4., wo M_1^4 in zwei irrat. verkn. M_1^2 oder in ein irrat. verkn. windschiefes Vierseit zerfällt.

11. $\infty^5 M_2^3$ mit M_1^1, M_1^2 windschief und Berührung in einem freien Punkte.

Classe IV. 1. ∞^5-$M_2^4\,(C_2^2)$ durch $M_1^4\,(p=1)$, die nicht zerfallen darf.

2. ∞^6-$M_2^4\,(C_2^2)$ durch $M_1^3\,(p=0)$, die nicht zerfallen darf.

3. ∞^7-$M_2^4\,(C_2^2)$ durch zwei windschiefe irrat. verkn. Geraden.

4. M_2^4 mit zwei irrat. verkn. Doppelgeraden nebst Curven wie n. 1. 2. 3.

Classe V. $\infty^3\,M_2^4\,(O^2)$ mit gemeinsamem Berührungsknoten O und gemeinsamer Tangentenebene in O nebst einer Basiscurve 12. O., Schnitt mit M_2^3.

In allen diesen Typen können typische Untersysteme durch Hinzunahme einfacher Basispunkte ausgeschieden werden, wo aber zu beachten ist, dass von n. 13 bis Ende der Classe I., dann in Classe II., in n. 4., 5., 9., 10. von Classe III. und in Classe IV kein rationaler Punkt unter den Basispunkten sein darf.

Auch wird man das Entstehen gemeinsamer Untertypen* in zwei verschiedenen Typen der obigen Aufzählung vermeiden können, indem man die neuen Basispunkte frei im Raume wählt (cf. auch die Anmerkung zu n. 2. der Classe I).

§10.—*Recapitulation unserer Methode. Das fundamentale Transversalentheorem für den R_3.*

Ueberblicken wir rasch nochmals den zurückgelegten Weg. Wir mussten uns zunächst Sicherheit verschaffen über diejenigen ∞^u-Systeme elliptischer Curven, welche auf Flächen existiren können, fanden diese entweder unicursal oder vom Kegelflächentypus, eruirten dann das Characteristische—nämlich die Trans-

* Die Zerlegung in M_1^4 $(p=0)$ mit Trisecante entsteht auch als Untersystem von No. 2, wenn man einen weiteren Basispunkt nicht frei, sondern auf einer Trisecante der M_1^4 von dort wählt.

† In der That wäre der folgende Complex n. 11 äquivalent einem Complexe n. 8, wo aber C_2 einen rationalen Punkt enthielte.

versalzahlen—jener Systeme, welche auf Flächen rational distinct ausgeschnitten werden können, setzten dann—oft nur summarisch—jene Characteristica für die typischen Systeme fest, machten dann nach §2 den Schluss, dass diese Transversalzahlen (und transversalen Curvensysteme) auch auf den M_2^n, welche Systeme elliptischer Curven von jenem Typus tragen, also auf jene typischen Flächen rational distinct abbildbar sind, wieder erscheinen müssen, hierauf jenen Schluss, dass sie auch in den eigentlichen und uneigentlichen Curvencomplexen wieder erscheinen müssen, machten dann nach §2 den weiteren Schluss, dass ausschneidende M_2^n-Systeme existiren müssen, für die wir eben aus ihren Transversaleigenschaften gegen die M_2^n den Rang 1 oder ihr Enthalten eines ∞^3-Systemes vom Range 1 herleiten konnten, womit wir sofort auf den Typus geführt werden.

Das Problem im R_3 ist damit gelöst, aber—und dies ist die Kraft der Transversalenmethode—mit einem Schlage auch das Problem im R_r. Hiezu bedarf es eines wichtigen Theoremes, das aus den bisherigen Discussionen extrahirt werden kann und nun ausgesprochen werden soll.

Um die erhaltenen Typen arithmetisch zu characterisiren, müssten wir für jeden einzelnen die sämmtlichen möglichen Transversalzahlen s und die Dimensionen der zugehörigen Transversalensysteme aufstellen. Aber für unseren Zweck ist hauptsächtlich ein System von Transversalen in's Auge zu fassen. Implicite haben wir bereits für die M_2^2- M_2^3-Typen davon Gebrauch gemacht, dass für die Curvencomplexe, zu denen sie führen, die Ebenen des R_3 ein rational-distinctes System bilden, während für die $M_2^4(O^2)$, $M_2^4(C_2^2)$,-$M_2^6(C_9^2)$-Typen die $M_2^2(O)$, $M_2^2(C_2)$ und $M_2^4(C_9)$ als rational distinctes System verwendet werden. Wird nun von den Typen zu den äquivalenten Curvencomplexen übergegangen, so bleiben diese Eigenschaften erhalten. Ja wir können sogar aussprechen:

THEOREM VIII.—*Für die Curvencomplexe erzeugt von M_2-Systemen vom monoidalen Typus gibt es ein rational distinctes ∞^2-System rationaler Curven, von denen durch jeden Punkt des R_3 nur eine Curve geht und welche die erzeugenden M_2 in nur einem Punkte treffen, aber ausserdem ein rational distinctes ∞^3-M_2-System Ranges* 1, *unter dessen gegenseitigen Schnittcurven jene enthalten sind.*

Für alle übrigen uneigentlichen Complexe und für die eigentlichen Complexe, von denen ein erzeugendes M_2-System den M_2^2-, M_2^3-, $M_2^4(C_2^2)$-Typus hat, gibt es ein rational distinctes ∞^3-M_2-System vom Range 1, *für die mit $M_2^2(C_2^2)$-Typus ausserdem ein rational distinctes ∞^3-M_2-System vom Range* 1, *für die mit $M_2^4(C_2^2)$-Typus ausserdem ein rational distinctes ∞^3-System vom Range* 2, *dagegen für die vom $M_2^4(O^2)$-Typus noch ein rat. dist. ∞^6-M_2-System vom Range* 4.

Bleibt noch zu beweisen, dass für die drei $M_2^4(C_2^2)$-Systeme das System der ∞^3 Ebenen ein rational distinctes Transversalensystem ist. Hiezu muss man sich der M_2^2 bedienen, welche resp. M_1^4, M_1^3, oder das Geradenpaar enthalten. Das System, das die R_2 auf diesen ausschneiden, hat den Rang 2, die Dimension 3; die Systeme, welche die übrigen ∞^3-Systeme vom Range 1, die unter den $\infty^4 M_2^2$ gewählt werden können, ausschneiden, haben stets, wenn man ihnen den Rang 2 verschafft, eine Dimension < 3. Nun sind jene M_2^2 rational distinct für das M_2^4-System, also sind auch die $\infty^3 R_2$ ein rational distinctes System.

Für $M_2^4(O^2)$ ist §13. al. 8. heranzuziehen.

§11.—*Einleitende Theoreme für die eigentlichen Complexe im R_r.*

Uns beschäftigt—bei eigentlichen Complexen—ein ∞^u-System von M_{r-1}, die sich zu je zwei in den Curven des $\infty^{(r-1)(u-r+2)}$-Complexes schneiden. Sie erzeugen also durch den Schnitt von je $r-2$ $\infty^{(r-2)(u-r+3)}$ Flächen, auf denen je ein ∞^{u-r+2}-System elliptischer Curven rational distinct enthalten sein wird. Diese Flächen werden von einem unserer Typen aus §§6, 7 sein und wir unterscheiden danach ebenso viele Classen von ∞^u-Complexen des R_r, entsprechend der Ebene, M_2^2, M_2^3, $M_2^4(C_2^2)$, $M_2^4(O^2)$.

Die M_3, welche innerhalb der M_{r-1}-Systeme erzeugt sind, sind, wie leicht zu sehen, unicursal. Denn nach §4 können wir schliessen, dass es ein ∞^t-System von M_{r-1} geben müsse, welches aus den M_3 die rational distincten ∞^t-M_2-Systeme ausschneiden, deren das Theorem VIII Erwähnung thut. Ist das rational distincte System ein homaloidales, so ist sogar die Complex-M_3 rational distinct auf den R_3 abbildbar. Uebrigens kann man auch das ∞^4-System vom Range 2 benützen und beim 5. Typus das ∞^6-System vom Range 4, und es bedarf dann nur der Festsetzung eines Punktes oder Punktetripels auf R_3, um ein System vom Range 1 auf M_3 festzusetzen, also ist mit Zuhilfenahme jener Irrationalitäten, welche durch das Schneiden mit einer Geraden eingeführt sind, die $M_3(1, 1)$-deutig auf einen R_3 abbildbar. Lassen wir einstweilen die $M_2^4(O^2)$-Classe bei Seite, die nur beim ∞^r-M_{r-1}-Systeme vorkommen kann.

Wir haben dann vermöge Theorem VIII. auf den erzeugten M_3 je ein rat. dist. ∞^3-System, das nun von einem rat. dist. ∞^3-M_{r-1}^m-System ausgeschnitten sein muss. Combiniren wir mit diesem ein ∞^{r-3}-System M_{r-1}^n und setzen beide in collineare Beziehungen zu zwei R_{r-1}-Systemen, resp. ∞^3 und ∞^{r-3}, in R_r', so erhalten wir eine birat. Transf., so dass die M_{r-1}^n gleichzeitig in solche übergeführt werden, welche von den R_{r-3} des ∞^3-Systemes in einem rat. dist. Systeme vom Range 1

geschnitten werden. Wird nun mittelst einer räumlich birationalen Transformation übertragen, welche die Systeme vom Range 1 in den $\infty^3 R_{r-3}$ in die R_{r-4}-Systeme in den R_{r-3} durch den festen R_{r-4} des R_r' verwandelt, so erhalten wir das Resultat:

THEOREM IX.—*Alle eigentlichen Complexe elliptischer Curven, die aus ∞^u-M_{r-1}-Systemen, $u > r-1$, erzeugt sind, sind räumlich birational übertragbar in solche, deren M_{r-1}^n einen festen $(n-1)$-fachen R_{r-4} besitzen, also Monoide mit R_{r-4}-Scheiteln sind.*

Die erzeugten Curven $p=1$ werden von diesem R_{r-4} auf den R_3 in die Curven eines Complexes transformirt. Wir können nun stets eine birationale Transformation einrichten, welche die R_{r-3} durch den R_{r-4} wieder in R_{r-3} durch den R_{r-4} verwandelt und ausserdem den eben genannten Curvencomplex in seinen Typus.

Somit entsteht ein System von R_{r-4}-Monoiden, deren gegenseitige Schnittcurven von R_{r-4} aus in Curven eines unserer typischen Complexe des R_3 projicirt werden.

Das Problem ist nun wieder so ungeformt, wie wir es in §6. aussprachen. Die Typen des §9 sind in die Abbildungen verschiedener Monoide zu situiren, und es ist zu fragen, durch welche Situirungen *verschiedene* Transversalzahlen des Complexes im R_r erreicht werden können.

Es bleibt noch der Beweis für die in das Theorem IX mit aufgenommene Classe der $M_2^4(O^2)$. Das System der ebenen Schnitte ist nun allerdings auf der einzelnen M_2^4 nicht rational distinct für das ∞^2-System der $C_4\, p=1$. Ist aber einmal die Curve 12. O. Schnitt mit M_2^3 hinzugenommen, welche Basiscurve der M_2^4 des ∞^2-Systemes ist, dann ist für das ganze ∞^3-System von M_2^4 dennoch das System der $\infty^3 R_2$ ein rational distinctes Transversalensystem. Wenn also ein bestimmtes ∞^r-System M_{r-1} im R_r vorliegt, dessen Schnitt-M_2 den $M_2^4(O^2)$-Typus haben, so werden die M_3 nothwendig, da sie ja nach Einleitung dieses § unicursal sind, ein für das rational distincte ∞^3-M_2-System in ihnen, also auch für sie selbst rational distinctes ∞^3-M_2-System vom Range 1 tragen. Damit haben wir denselben Ausgangspunkt wie vorhin für die anderen Classen gewonnen und schliessen wieder wie für die anderen Classen.

Auch hier kann stets birational so transformirt werden, dass die Projection des Complexes von R_{r-4} aus auf den R_3 der Typus im R_3 wird.

Was nun zuerst die Classe I. aus §9 betrifft, so haben wir diesfalls sogar auf den M_2 ein rat. dist. System vom Range 1; es existiren also ∞^2 M_{r-2}, welche die

M_2 des Complexes in einzelnen Punkten schneiden. Gehen wir genau so vor, wie vorhin, so erhalten wir:

THEOREM X.—*Alle eigentlichen Complexe von Curven $p = 1$ im R_r, deren M_2 vom Ebenentypus sind, sind in rational distincter Weise räumlich birational solchen Complexen äquivalent, welche innerhalb eines Systemes von R_{r-3}-Monoiden erzeugt sind.*

Bemerken wir, dass wir im R_3 Systeme von M_2^2, M_2^3, M_2^4 als Typen haben und Systeme von M_2^5, M_2^6 nur bei der Classe I. und nur bei Geltendmachung irrationaler Verknüpfungen unter gewissen Geradengruppen. Wir können nun in der Abbildung die M_2^2-Systeme schon bei M_{r-1}^2-, die M_2^3-Systeme schon bei M_{r-1}^3-, die M_2^4-Systeme schon bei M_{r-1}^4-Systemen zu vollständigen Schnitten ergänzen.

Hiezu ist es nützlich, über die Abbildung der M_{r-1}^2 (R_{r-3}), M_{r-1}^3 (R_{r-4}^2), M_{r-1}^4 (R_{r-3}^3), M_{r-1}^4 (R_{r-4}^3) einiges vorauszuschicken.

1. *Abbildung der M_{r-1}^2 (R_{r-3})* auf den R_{r-1}^b geschieht durch Projection aus einem Punkte O des R_{r-3}. Bild des O wird ein $R_{r-2} \equiv \Omega$, welcher durch den Schnitt-R_{r-4} des R_{r-3} mit R_{r-1}^b geht. Jeder R_{r-2} durch den R_{r-3} schneidet M_{r-1}^2 in einem R_{r-3}, der den ersten in R_{r-4} trifft. Dieser gibt es ∞^2, die also einen R_{r-6} gemeinsam haben. M_{r-1}^3 enthält diesen R_{r-6} zweifach, ist also ein R_{r-6}-Kegel. Bilder der Schnitte von M_{r-1}^2 mit allen R_{r-1} des R_r sind M_{r-2}^2 durch eine feste M_{r-3}^2 in Ω, die O_ω heissen soll. O_ω enthält den R_{r-4}^2 einfach. Bilder der Schnitte von M_{r-1}^3 mit M_{r-1}^2 durch O sind M_{r-2}^3 (O_ω). Diese M_{r-2}^3 besitzen also im R_{r-4} eine Doppel-M_{r-7}^3, welche aber variabel ist. Bilder der Schnitte mit M_{r-1}^2 durch R_{r-3} sind M_{r-2}^3-$(R_{r-4}^2)^2$, welche O_ω einfach enthalten. Wenn diese M_{r-1}^2 in O die gegebene M_{r-1}^2 berühren, so haben die Schnitte zu Bildern M_{r-2}^2 (R_{r-4}), welche O_ω nicht enthalten und also O_ω ausser im R_{r-4} noch in einer M_{r-4}^3 schneiden. Wenn diese M_{r-2}^2 (R_{r-4}) die Ω längs des R_{r-4} berühren, so tritt das in n. 1 der Aufstellung des Theoremes X. beschriebene Verhalten ein.

2. *Abbildung der M_{r-1}^3 (R_{r-3}^2)* auf den R_{r-1}^2 geschieht durch Projection aus einem Punkte O des R_{r-3}. Bild des O wird eine $M_{r-2}^2 \equiv \Omega$, welche durch den Schnitt-R_{r-4} des R_{r-3}^2 mit R_{r-1}^b doppelt geht.—Jeder R_{r-2} durch den R_{r-3} schneidet M_{r-1}^3 in einem R_{r-3}, der den ersten im R_{r-4} trifft. Dieser gibt es ∞^2, die also einen R_{r-6} gemeinsam haben, der für M_{r-1}^2 dreifach ist, also ist M_{r-1}^3 von $r = 6$ ab im R_{r-6}-Kegel. Im linearen M_{r-1}^3-Systeme variirt der Scheitel R_{r-6}.—Bilder der Schnitte von M_{r-1}^3 mit allen R_{r-1} des R_r sind M_{r-2}^3 durch eine

feste M^6_{r-3} in Ω, die O_ω heisse. O_ω enthält R_{r-4} 4-fach und ist der Schnitt der von O aus in die M^3_{r-1}gehen den Geraden auf dem R^b_{r-1}. Bilder der Schnitte mit M^3_{r-1} sind M^3_{r-2} durch O^3_ω, der Schnitte mit $M^3_{r-1}(O^2)$ sind M^5_{r-2} durch O_ω und mit R^3_{r-4}, der Schnitte mit $M^3_{r-1}(R^2_{r-3})$, sind $M^5_{r-2}(R^4_{r-2})$ durch O_ω, der Schnitte mit $M^3_{r-1}(R^2_{r-3})$ welche M^2_{r-1} in O berühren, sind $M^3_{r-2}(R^2_{r-4})$, welche O_ω nicht enthalten müssen. Der Ausdruck, "berühren in O" bedeutet, dass beide M^3_{r-1} in O dieselben R_2 gestatten, in denen die Schnittcurve $M^3_1(O^2)$ in O eine Spitze besitzt. Die Spitzentangenten schneiden auf Ω; zu ihnen gehören specieller Weise die Geraden von O, welche in O_ω schneiden. Jede Ebene durch O schneidet die beiden M^3_{r-1} in zwei $M^3_1(O^2)$ mit übereinstimmenden Doppelpunktstangenten.

Im Allgemeinen gibt es für die beiden $M^3_{r-1}(R^2_{r-3})$ einen biquadratischen Kegel von gemeinsamen Tangenten—R_{r-2} durch R_{r-3}. Wenn aber die $M^3_{r-2}(R^2_{r-4})$—die Schnittbilder—die Ω längs des R_{r-4} berühren, dann reducirt sich der biquadratische Kegel auf einen doppelt gezählten quadratischen. Um eine Vorstellung von diesem Verhalten zu geben, will ich die $M^3_2(R^2_1)$ des R_3 auf die Ebene abbilden. Die Schnitte mit anderen $M^3_2(R^2_1)$ bilden sich als $M^5_1(R^4_0)$ ab und wenn sich die zwei Bildgeraden von O absondern, noch durch $M^3_1(R^2_0)$. Diese sind Bilder von M^5_1, welche O^2 enthalten und R_1 noch zweimal variabel und, wenn die $M^3_1(R^2_0)$ feste Doppelpunktstangenten hat, mit festen Tangentenebenen durch R_1 treffen und aber, wenn $M^3_1(R^2_0)$ jene zwei Bildgeraden berühren, diese variabeln Schnittpunkte mit R_1 in O selbst hineinziehen. Es fallen also von den 4 durch R_1 gehenden Tangentenebenen der Schnittcurve M^5_1 je zwei in eine zusammen, nämlich in die beiden Tangentenebenen des Punktes O. So ist es also—mit entsprechender Aenderung der Dimension—zu verstehen, wenn im §12 in der Aufzählung steht: "mit einem doppelt gezählten quadratischen Kegel gemeinsamer Berührungs-R_{r-2}."

3. *Abbildung der* $M^4_{r-1}(R^3_{r-4})$ auf den R^b_{r-1} geschieht wieder durch Projection aus einem Punkte O der R_{r-4}: Bild von O wird eine $M^3_{r-2}(R^4_{r-5})^3 \equiv \Omega$, wo R_{r-5} der Schnitt des R_{r-4} mit R^b_{r-1} ist. Jeder R_{r-3} durch den R_{r-n} schneidet M^3_{r-1} in einem R_{r-4}, der den ersten in einem R_{r-5} trifft. Dieser gibt es ∞^3, die also einem R_{r-8} gemeinsam haben, der für M^4_{r-1} einfach ist, also ist M^4_{r-1} von $r=8$ ab ein R_{r-8}-Kegel. Im linearen M^4_{r-1}-Systeme variirt aber dieser Scheitel R_{r-8}.

Bilder der Schnitte von M^4_{r-1} mit allen R_{r-1} des R_r sind $M^4_{r-2}(R^3_{r-5})$ durch eine feste M^{12}_{r-3} in Ω, die wieder O_ω heisse. O_ω enthält R_{r-5} 9-fach und ist der

Schnitt der von O aus in die M^4_{r-1} gehenden Geraden auf dem R^5_{r-1}. Bilder der Schnitte mit M^4_{r-1} sind M^{16}_{r-2} durch O^4_ω, der Schnitt mit $M^4_{r-1}(O^3)$ sind M^7_{r-2} durch O_ω und mit R^3_{r-1}, mit $M^4_{r-2}(R^3_{r-4})$ sind $M^7_{r-2}(R^6_{r-5})$ durch O_ω und wenn in O Berührung stattfinden soll, also die Tangentenebenen der beiden M^4_{r-2} in O übereinstimmen sollen, werden die Schnittbilder $M^4_{r-2}(R^3_{r-5})$ sein. Zwei solche M^4_{r-1} werden von jeder Ebene durch O in zwei Curven $M^4_1(O^3)$ geschnitten, die im dreifachen Punkte übereinstimmende Tangenten besitzen. Wenn aber die letzterwähnten M^4_{r-2} die Ω längs des R_{r-5} berühren, so tritt ein ähnliches Verhalten wie bei $M^3_{r-1}(R^2_{r-4})$ ein und die Beschreibung in Classe IV und V des §12 ist daher in derselben Weise durch die Abbildung zu commentiren.

§12.— *Ueber die Aufstellung der Typen im R_r nach meiner Methode des* §4.

M^2_{r-1}. Gehen wir zunächst zum R_4 über und suchen ein vollständiges Schnittsystem von $M^2_3(R_1)$ zu ergänzen, ausgehend von einem typischen Complexe des §9. Nur die Systeme von M^2_2 lassen sich derart situiren und hierzu muss Ω doppelt abgerechnet werden, was auf $M^2_3(R_1)$ führt, die sich in einem Punkte O_1 von (R_1) berühren. Von hier zum R_5 übergehend verlegen wir diesen Punkt O_1 des Bildsystemes auf O_ω, lassen die feste Berührungsebene mit der von O_ω coincidiren und verlegen auch den festen Punkt O aus dem R_3 her auf O_ω und können nicht O gleichzeitig in eine Gerade aus O verlegen, da sonst eine Gerade sich von den Schnittcurven absondern würde. Setzen wir so weiter bis zum R_r fort, so erhalten wir die in n. 1, Classe I. hier folgend beschriebene Berührung.

M^3_{r-1}. Indem wieder mit R_4 begonnen werde, versuchen wir, ein System von $M^3_3(R^2_1)$ herzustellen, dessen einzelne M^3_3 durch die Abbildung n. 2 aus §11 den Typus 2. der Classe I. aus §9 liefere. Von den $M^5_2(R^4_0)$ können wir die $M^3_2(R^2_0)$ des Typus § 9 absondern und es bleibt als Ergänzung $M^3_2(R^2_0)$. Da die M^3_2 des genannten Typus nicht durch O_ω hindurch gehen können, so muss Ω selbst als Ergänzung genommen werden. Nehmen wir überdies den festen Osculationskegel des Typus als mit dem Kegel Ω übereinstimmend, so handelt es sich nun, die Lage des $M^3_3(R^2_1)$-Systemes im R_4 zu definiren, dessen Schnitte mit der gegebenen M^3_3 deren Typus als Bild liefern. Wir sahen aber in der Abbildung n. 2., dass in diesem Falle der sonst biquadratische Kegel der gemeinsamen Tangentenebenen um R_1 in einen doppelt gezählten quadratischen ausartet. Da nun der Osculationskegel von Ω in R_0 stets auch im R_6, ... bis R_r quadratisch ist, so können wir stets den Osculationskegel aus dem Typus im R_{r-1} mit dem von Ω zusammenfallen lassen und erhalten stets einen Typus gleicher Art.

Den Typen 3. bis 7. der Classe entsprechende Typen finden wir im R_4 bis R_r dadurch—ebenso wie wir sie im R_3 entdeckten—dass wir die M^2_{r-2}, durch welche wir die M^3_{r-2} zum vollständigen Schnittbilde M^5_{r-2} ergänzen müssen, einmal mit Ω zusammenfallen lassen, wo es dann noch für die Transversalzahlen im R_r, also für die Möglichkeit, eine birationale Aequivalenz wie in §4 einzurichten, also auch für die Natur des Complexes indifferent ist, ob wir die Basis des Bildcomplexes in die Ω verlegen oder frei im R_r lassen, ein anderes Mal aber die Ergänzungs-M^2_{r-2} von Ω verschieden nehmen, zu diesem Zwecke aber die Basis des Bildcomplexes im R_{r-1} auf Ω selbst und zwar gerade als Bestandtheil von O_ω wählen müssen, damit der für die vollständigen Schnitte geforderte Durchgang durch O_ω geleistet sei. Durch den restlichen Theil von O_ω geht die Ergänzungs M^2_{r-2}, der als Bild denn auf M^3_{r-1} im R_r eine Original-M_{r-2} entspricht, welche eben in den Fällen n. 4 bis 7. bezüglich eine M^3_{r-2}, M^4_{r-2}, M^5_{r-2}, M^6_{r-2} ist. Ebenso verhält es sich mit den übrigen Typen der Classe I. aus §9.

Bei der Classe III. tritt von R_4 angefangen eine grosse Verschiedenheit gegenüber dem R_3 auf, die ihren Ursprung wiederum in der Ergänzungsmannigfaltigkeit zu dem den Bildcomplex erzeugenden Systeme hat und die ich erläutern muss. Wir haben es im R_4 mit $M^3_3(R^2_0)$ zu thun. In Abbildung n. 3. aus §13 haben wir als O_ω eine Curve $M^6_1 p = 4$ und müssen im Bild-R_3 die Complexe der Classe III. aus §9 situiren. Der erste hat die Basis $M^5_1 p = 2$; es ist auch thatsächlich möglich, eine M^3_3 zu construiren, wo O_ω sich in $C_5 p = 2$ und eine Trisecante spaltet. Durch diese Gerade wird die Ergänzungs-M^2_2 hindurchgehen müssen und wir können sie nun 1. mit Ω coincidiren lassen oder sie 2. davon verschieden nehmen, sodass zwei verschiedene Complexe im R_4 entstehen:

1. $M^3_3(R^2_0)$ mit fester Osculation im R_0 und einem im Osculationskegel enthaltenen $M^5_2(R^5_0)$.

2. $M^3_3(R^2_0)$ mit einer Basis-$M^5_2(R_0)$.

Gehen wir nun von n. 1. zu R_5 über, so werden wir im Bild-R_4 diese $M^3_3(R^2_0)$ so verlegen, dass der Doppelpunkt mit R_0 coincidirt und dass ausserdem der Osculationskegel den wir oben eruirt haben, mit dem Osculationskegel von Ω, welche noch Abbildung 2. des §11 $M^2_2(R^2_0)$ ist, coincidirt. Wir bekommen so von R_5 bis R_r stets nur denselben Typus (bis auf die Dimension). Gehen wir aber von n. 2. ab weiter, so haben wir, da dieser Typus bis auf die Dimension genau mit n. 1 der Classe III. im R_3 übereinstimmt, allerdings neuerdings die Möglichkeit zweier Situirungen. Aber die eine derselben liefert immer wieder nur den oben gefundenen Typus, die andere eben den dem 2. im R_1 entsprechenden

Typus. So verhält es sich beinahe mit jedem Typus der Classe III. und wir haben vom R_4 ab die doppelte Anzahl Typen jener im R_3, mit Ausnahme nur des Typus mit 3 Geraden. Denn dann ist der Rest von O_ω wieder ein windschiefes Geradentripel, und da die Ergänzung dieses enthalten muss, fällt sie *nothwendig* mit Ω zusammen.

Classe III liefert aber noch eine weitere neue Besonderheit gegenüber dem R_3. Nämlich die in R_3 scheinbar nebensächlichen Degenerationen der Basiscurven führen hier dadurch, dass man nur einen Theil derselben in die O_ω eingehen lässt, den anderen ausserhalb nimmt, zur Möglichkeit, neue Ergänzungs-M_2 verschieden von Ω zu verwenden. So sind einige im R_r neue Typen der Classe III entstanden.

Classe IV. bietet gegen Classe III. einige Eigenthümlichkeiten. Gemäss Abbildung n. 3. aus §11 haben wir für R_4 die $M_2^4(C_2^2)$ des Bild-R_3 zu ergänzen zu M_2^7. Dies kann in zwei Arten geschehen. Wir verlegen C_2 und resp. M_1^4, M_1^3, $M_1^1 + M_1^1$ in Ω, indem wir sie sogar als Bestandtheil von O_ω wählen, welche der Schnitt 12. O. einer M_2^2 und einer M_2^3 sein muss. An einer Hilfsabbildung von M_2^3 erprobt man, dass der Resttheil, der von O_ω nach Wegnahme von $C_2 + M_1^1 + M_1^1$ bleibt, nicht mehr das Hindurchgehen einer zweiten M_2^3 gestattet. Nach Wegnahme von $C_2 + M_1^4$ oder $C_2 + M_1^3$ bleibt dagegen M_1^6 oder M_1^7, durch welche ausser Ω eine zweite M_2^3 als die nothwendige Ergänzungsfläche gehen kann. Da die Bild-M_2^4 die C_2 doppelt enthalten, müssen wir einen Kegel $M_2^2(O^2)$ hinzufügen,* der C_2 enthält, und aus der Ergänzung entsteht resp. eine $M_2^6(O^3)$ oder $M_2^5(O^2)$, welche die Eigenschaft besitzt, in einer M_2^3 zu sein. Für Ω als Ergänzung entsteht dagegen der feste Osculationskegel in O und ein Berührungskegel $M_2^2(O^2)$ selbst $M_2^4(O^4)$ oder $M_2^3(O^3)$ oder $M_2^1(O) + M_2^1(O)'$, welche alle in Osculationskegel enthalten sind.

Weitergehend zum R_5 haben wir also im Bildraume R_4^b einen der beiden im R_4 enthaltenen typischen Complexe zu situiren. Für die drei Typen mit festem Osculationskegel lassen wir überdies die drei Basiskegel $M_2^2(R_0^2) + M_2^4(R_0^4)$ oder $M_2^2(R_0^2) + M_2^3(R_0^3)$ oder $M_2^2(R_0^2) + R_1(R_0) + R_1^1(R_0)$ in die O_ω, welche jetzt eine $M_2^{12}(R_0^9)$ Schnitt vom $\Omega \equiv M_3^3$ und M_3^4 ist, als Theile eingehen. In Folge davon wird der Osculationskegel aller Bild M_3 nothwendig durch diese Basiskegel hindurchgehen und nichts hindert, ihn mit dem Osculationskegel Ω selbst coin-

* Dieses Verfahren drückt sich arithmetisch durch die Thatsache aus, dass man für einen Punkt A, der a-fach gezählt ist, den Singularitätencomplex $n = 0, -a, 0 \ldots . 0$, wo die letzten Nullen sich auf die etwa weiter vorhandenen Grundpunkte beziehen. Cf. meine Arbeit in Cr. J., CXIV.

cidiren zu lassen. Wir bekommen damit die in n. 3. des §11 erwähnte Berührungsart, wenn wir Ω als Ergänzung nehmen. Aber wie im R_4 bekommen wir aus den beiden ersten die Möglichkeit, die Ergänzungs-M_3^3 durch den Resttheil von O_ω verschieden von Ω situiren. Diese liefert dann im R_5 einen Typus, wo nur der der C_2 entstammende M_3^2-Kegel erscheint und ausserdem eine $M_3^6(R_1)$ oder $M_3^5(R_1)$.

Es bleiben noch zwei Bilder zu verwenden, die beiden letzten Typen aus dem R_4. Da dort die Basis-M_2^6,-M_2^5 in einer M_3^3 enthalten waren, können wir sie in O_ω als Theile eingehen lassen. Dagegen ist es nicht möglich, gleichzeitig den M_2^2-Kegel in dieselbe O_ω mit aufzunehmen, aber wir können ihn so situiren, das ihm auf den M_4^4 des R_5 nicht ein Kegel 4. O., sondern nur ein Kegel 2. O. entspricht (Typen n. 6., 7. der Classe IV). Die Ergänzungs-M_3^3 durch die restlichen M_2^6, M_2^7 liefert im R_5 wieder M_3^6, M_3^5 derselben Art bis auf die Dimension und so entsteht also eine ganze Reihe von Typen bis zum R_r, welche in der Tafel unter Cl. IV erscheint.

Wird aber mit jenen zwei letzten Bildern aus dem R_4 doch noch die Ergänzungs-M_3^3 mit Ω identisch gemacht, so entsteht zunächst im R_5 ein System von M_4^4 mit fester Osculation nicht nur in O sondern so, dass die gemeinsamen Tangentenebenen einen doppelt gezählten quadratischen Kegel bilden, und einen gemeinsamen Kegel M_3^6, M_3^5 und ausserdem wiederum dem Basiskegel 2. O. M_2^2. Setzt man von diesem Typus im Bild-R_5 angefangen dieses Verfahren bis zum R_r fort, so erhält man immer wieder einen Typus derselben Art oder einen Typus der schon vorhin erhaltenen beiden Arten.

Indem ich nun die soeben in einzelnen Fällen angedeutete Anwendung meiner Methode durchgeführt denke, erhalte ich folgendes Theorem, das ich jedoch mit dem Vorbehalte hier ausspreche, dass sich bei späterer Discussion eine Subdivision der Typen herausstellen könne.

THEOREM XI.—*Alle eigentlichen Complexe elliptischer Curven im R_r, welche durch ∞^u-M_{r-1}-Systeme, $u > r-1$, erzeugt sind, können in rational distincter Art räumlich birational äquivalent einem der folgenden Typen gemacht werden:*

Classe I. 1. $\infty^{r+\sigma}M_{r-1}^2$, welche einen rational distincten R_{r-3} gemeinsam haben, und mit einem doppelt zu zählenden R_{r-1} von gemeinsamen Berührungs-R_{r-2} durch einen R_{r-4} im R_{r-3} nebst ausserdem einer Anzahl einfacher Basispunkte, die frei im R_r gewählt werden können.

2. $\infty^{r-\alpha+\tau}M_{r-1}^3(R_{r-3}^2)$, welche längs dieses R_{r-3} gemeinsame Tangenten-R_{r-2} in einem doppelt gezählten quadratischen Kegel besitzen und aus-

serdem $\alpha = 0, 1, \ldots . 6$ gemeinsame Punkte, die verschiedentlich irrational verknüpft sein können.

3. $\infty^{r+4} M^2_{r-1}(R^2_{r-3})$ mit gemeinsamer $M^3_{r-2}(R_{r-3})$.

4. $\infty^{r+3} M^3_{r-1}(R^2_{r-3})$ mit gemeinsamer $M^4_{r-2}(R^2_{r-3})$.

5. $\infty^{r+2} M^3_{r-1}(R^2_{r-3})$ mit gemeinsamer $M^5_{r-2}(R^3_{r-3})$.

6. $\infty^{r+1} M^3_{r-1}(R^2_{r-3})$ mit gemeinsamer $M^6_{r-2}(R^4_{r-3})$.

7. $\infty^{r+1} M^3_{r-1}(R^2_{r-3})$ mit gemeinsamem Tangentenkegel wie in n. 2. mit 6 einfachen freien Punkten, welche irrational verknüpft sind und nicht in einem quadratischen Kegel mit R_{r-3} als Scheitel enthalten sind.

8. $\infty^{r+1} M^3_{r-1}(R^2_{r-3})$ mit derselben Berührung um den R_{r-3} wie in n. 2. und mit mehr als zwei einfachen Punkten, unter denen aber auch Tripel, die mit R_{r-3} in einem R_{r-1} sind, oder Berührungen vorhanden sein können.

9. $\infty^{r+1} M^3_{r-1}(R^2_{r-3})$ mit gemeinsamem Osculationsraume R_{r-1} durch den R_{r-3} und einer gemeinsamen $M^3_{r-2}(R^2_{r-3})$, die aber nicht in jenem R_{r-1} enthalten sind.

10. $\infty^{r+1} M^3_{r-1}(R^2_{r-3})$ mit zwei $M^3_{r-2}(R^2_{r-3})$,* die in zwei R_{r-1} durch R_{r-3} enthalten sind und deren Osculationskegel in einen und denselben doppelt gezählten R_{r-2} ausgeartet ist.

11. $\infty^{r+1} M^4_{r-1}(R^3_{r-3})$ mit gemeinsamer $M^{10}_{r-2}(R^6_{r-3})$.

12. $\infty^{r} M^4_{r-1}(R^3_{r-3})$ mit gemeinsamer $M^{11}_{r-2}(R^9_{r-3})$.

[Hieher gehört das ∞^{r-1}-System $M^4_{r-1}(R^3_{r-3})$ mit gemeinsamer $M^{12}_{r-2}(R^{10}_{r-3})$].

13. $\infty^{r+4} M^3_{r-1}(R^2_{r-3})$ mit zwei irrational verknüpften R_{r-2} durch R_{r-3} und einem R_{r-2}, der nicht durch den R_{r-3} geht.

14. $\infty^{r+2} M^3_{r-1}(R^2_{r-3})$ mit zwei R_{r-2} durch R_{r-3} und $M^3_{r-2}(R^2_{r-3})$.

15. $\infty^{r+\sigma} M^4_{r-1}(R^3_{r-3})$ mit der zu n. 2. analogen Berührung um R_{r-3} längs eines doppelt gezählten cubischen Kegels und mit Berührung längs zweier R_{r-2} darunter.

16. $\infty^{r+4} M^4_{r-1}(R^3_{r-3})$ wie 15. nebst fester $M^6_{r-2}(R_{r-3})$.

17. $\infty^{r+3} M^4_{r-1}(R^3_{r-3})$ wie 15. nebst fester $M^5_{r-2}(R^3_{r-3})$ und irrationaler Verknüpfung der R_{r-2}.

18. $\infty^{r+2} M^4_{r-1}(R^3_{r-3})$ mit zwei irrational verknüpften R_{r-2} durch R_{r-3} und $M^6_{r-2}(R^3_{r-3})$.

19. $\infty^{r+1} M^4_{r-1}(R^3_{r-3})$ mit zwei R_{r-2} durch R_{r-3} und $M^7_{r-2}(R^4_{r-3})$.

20. $\infty^{r} M^4_{r-1}(R^3_{r-3})$ mit zwei R_{r-2} durch R_{r-3} und $M^8_{r-2}(R^5_{r-3})$.

---

* Diese M^3_{r-2} sind jedoch sehr specieller, näher zu beschreibender Natur.

21. $\infty^{r+3} M^4_{r-1}(R^3_{r-3})$ mit 5 irrational verknüpften R_{r-2} durch den R_{r-3} und $M^4_{r-2}(R^2_{r-3})$.

22. $\infty^{r+3} M^5_{r-1}(R^4_{r-3})$ mit der zu n. 2. analogen Berührung um R_{r-3} längs eines doppelt gezählten biquadratischen Kegels und 5 irrational verknüpften R_{r-2} durch den R_{r-3}.

23. $\infty^{r+1} M^5_{r-1}(R^4_{r-3})$ mit $M^4_{r-2}(R_{r-3})$ und Berührung längs dreier, Osculation längs zweier je irrational verknüpften R_{r-2} durch den R_{r-3}.

24. $\infty^{r+3} M^5_{r-1}(R^4_{r-3})$ mit $M^{11}_{r-2}(R^5_{r-3})$ und 5 irrational verknüpften R_{r-2} durch den R_{r-3}.

25. $\infty^{r+2} M^5_{r-1}(R^4_{r-3})$ mit $M^3_{r-2}(R_{r-3})$ und mit R_{r-3} wie in 24.

26. $\infty^{r+4} M^6_{r-1}(R^5_{r-3})$ mit R_{r-2} wie in 24.

Classe II. $\infty^u M^2_{r-1}$ mit Berührung wie in Classe I. n. 1. aber ohne den gemeinsamen R_{r-3} und Gruppen gemeinsamer Punkte, unter denen aber keiner rational gegeben sein darf.

Classe III. 1. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit gemeinsamem Tangenten-R_{r-3}-Kegel welcher ein doppelt gezählter quadratischer ist, und einem unter diesem enthaltenen Kegel $M^5_{r-2}(R^5_{r-4})$.

2. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit einer Basis $M^5_{r-2}(R^3_{r-4})$.

3. $\infty^{r+4} M^3_{r-1}(R^2_{r-4})$ mit Berührung wie in n. 1. und einem Basiskegel $M^4_{r-2}(R^4_{r-4})\, p = 0$, 2. Art.

4. $\infty^{r+4} M^3_{r-1}(R^2_{r-4})$ mit einer Basis-$M^4_{r-2}(R^2_{r-4})$, $p = 0$, 2. Art.

5. $\infty^{r+4} M^3_{r-1}(R^2_{r-4})$ mit Berührung wie in n. 1. und drei gemeinsamen R_{r-2} durch den R_{r-4}.

6. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit Berührung wie in n. 1. und gemeinsamen Kegel $M^2_{r-2}(R^2_{r-4})$, gemeinsamen $R_{r-2}(R_{r-4})$, der jenen nicht anormal schneidet und Berührung längs gemeinsamem R_{r-3} durch R_{r-4}.

7. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit Berührung wie in n. 1. und gemeinsamem Kegel $M^4_{r-2}(R^4_{r-4})\, p = 1$ und Berührung längs gemeinsamem R_{r-3} durch R_{r-4}.

8. $\infty^u M^3_{r-1}(R^2_{r-4})$-Systeme mit Berührung wie in n. 1. und gemeinsamen Berührungs-R_3 in 2, 3, 4 irrational verknüpften freien Punkten der R_r.

9. Degenerationen, welche aus n. 1. durch Zerfallen der M^5_{r-2} in M^4_{r-2} mit R_{r-2}, oder

10. in zwei irrational verknüpfte M^2_{r-2} und einen beide anormal schneidende R_{r-2} oder

11. eine in einem $R_{r-1}(R_{r-4})$ enthaltene M^3_{r-2} und eine in einer M^2_{r-3} schneidende M^2_{r-2} entstehen, jedoch mit dem Vorbehalte, dass man auf keiner

dieser Mannigfaltigkeiten einen rationalen Punkt kenne. Bei 11. muss der Tangentenkegel um den R_{r-4} in zwei R_{r-1} zerfallen.

12. Degeneration, welche aus n. 3. durch Zerfallen von M^4_{r-2} in zwei sich in einem R_{r-3} schneidende M^2_{r-2} entsteht.

13. Degeneration von n. 6., wo M^2_{r-2} in ein rational verknüpftes R_{r-2}-Paar zerfällt.

14. Degeneration von n. 7., wo M^4_{r-2} in zwei irrational verknüpfte M^2_{r-2} oder ein irrational verknüpftes Quadrupel von R_{r-2}, die aber keinen gemeinsamen R_{r-3} bestimmen, zerfällt.

15. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit R_1, M^2_{r-2} und gemeinsamen Berührung-R_2 in einem freien Punkte des R_r (parallel mit n. 6).

16. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit zwei sich in einem R_0 schneidenden Variatäten M^2_1, M^2_{r-2} (parallel mit n. 10).

17. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit $M^4_{r-2}(R^2_{r-4})$ mit einer diese zweimal schneidenden Geraden.

18. System wie n. 17., ausser dass jene M^4_{r-2} in zwei $M^2_{r-2}(R_{r-4})$ zerfällt (parallel mit 10.).

19. $\infty^{r+3} M^3_{r-1}(R^2_{r-4})$ mit gemeinsamem Tangenten-R_{r-3} in einem R_{r-1} um den R_{r-4} und im R_{r-1} einer festen M^2_{r-2} und ausserhalb des R_{r-1} einer festen $M^3_{r-2}(R^2_{r-4})$.

Classe IV. 1. $\infty^{r+2} M^4_{r-1}(R^2_{r-4})$ mit gemeinsamer Berührung längs eines doppelt gezählten cubischen Kegels um den R_{r-4} und mit einer Basis-$M^4_{r-2}(R^4_{r-4})\ p=1$, die nicht zerfallen darf, und welche aus Tangenten-R_{r-3} um den R_{r-4} besteht, nebst Berührung längs einer Basis-$M^2_{r-2}(R^2_{r-4})$.

2. $\infty^{r+3} M^4_{r-1}(R^3_{r-4})$ mit Berührung wie in n. 1. und einer Basis-$M^3_{r-2}(R^3_{r-4})\ p=0$, die nicht zerfallen darf, und mit Berührung längs einer Basis-$M^2_{r-2}(R^2_{r-4})$.

3. $\infty^{r+4} M^4_{r-1}(R^3_{r-4})$ mit Berührung wie in n. 1. und zwei Basis-$R_{r-2}(R_{r-4})$, die sich nicht normal schneiden, und mit Berührung längs einer Basis-$M^2_{r-2}(R^2_{r-4})$.

4, $\infty^{r+2} M^4_{r-1}(R^3_{r-4})$ mit gemeinsamer $M^2_{r-2}(R^2_{r-4})$ und einer Basis-$M^6_{r-2}(R^3_{r-4})$.

5. $\infty^{r+3} M^4_{r-1}(R^3_{r-4})$ mit gemeinsamer $M^2_{r-2}(R^2_{r-4})$ und einer Basis-$M^5_{r-2}(R^2_{r-4})$.

6. $\infty^{r+3} M^4_{r-1}(R^3_{r-4})$ mit Berührung wie in n. 1., einer Basis-M^6_{r-2} und einer gemeinsamen M^2_2.

7. $\infty^{r+2} M^4_{r-1}(R^3_{r-4})$ mit Berührung wie in n. 1., mit einer Basis-M^6_{r-2} und einer gemeinsamen M^2_2.

8. Degenerationen, welche aus n. 1.–7. entstehen, indem die vorkommenden quadratischen Kegel in rational verknüpfte Paare linearer Räume zerfallen.

Classe V. $\infty^r M^4_{r-1}(R^3_{r-4})$ mit einer Selbstberührung in einem R_{r-4} und einem gemeinsamen Basiskegel M_{r-2} 12. Ordnung.

Hiezu kommen typische Untersysteme, welche aus den vorhergehenden durch Festsetzung gemeinsamer einfacher freier Punkte des R_r entstehen, wobei aber zu beachten ist, dass, um irreductible Systeme zu erhalten, unter diesen Punktgruppen in Classe I. von 13. bis 25., in Classe II und in Classe IV. kein einziger rational gegebener Punkt sein darf. Ob noch weitere Einschränkungen zu machen sind, kann ich derzeit nicht entscheiden, da hiezu eine Discussion aller verwendbaren Raumtranspositionen nöthig wäre.

§13.—*Die uneigentlichen Complexe elliptischer Curven im R_r.*

Jeden linearen Complex von der Art der in dieser Arbeit untersuchten, also in Bezug auf Dimension vollständigen, sei er eigentlich oder uneigentlich, können wir durch $r-1$ lineare M_{r-1}-Systeme von verschiedenen oder gleichen Dimensionen erzeugen.* Von diesen erzeugen $r-2$ einen vollen M_2-Complex. Wählen wir sie so, dass das $(r-1)$. System eine Dimension >1 hat, was möglich sein wird, da, wenn alle $r-1$ M_{r-1}-Systeme gleiche Dimension 1 hätten, der Complex eigentlich wäre und durch ein einziges M_{r-1}-System erzeugt werden könnte, so wird auf jeder M_2 des genannten Complexes ein mindestens ∞^2-System vorhanden sein; also werden die M_2 abbildbar sein; ja, da der Complex rational distinct gegeben ist, werden diese Curvensysteme auf den M_2 rational distinct und die M_2 werden von einem der Typen des Theoremes VII. sein. Nur wenn M_2 den Kegelflächentypus hätte, müssten wir zuerst bedenken, dass die sämmtlichen Schaaren rationaler Curven in einem einzigen ∞^{r-1}-Systeme enthalten sein müssen, wo von wir auf das Strahlbündel und damit auf den Typus des §3 kommen. Setzen wir also voraus, dass es nicht möglich sei, $r-2$ unter den $r-1$ Systemen so auszuwählen, dass sie M_2 vom Kegelflächentypus erzeugen.

Dann wählen wir $r-3$ unter den Systemen. Sie erzeugen einen M_3-Complex und diese M_3 werden nach den in §11 gemachten Schlusse wieder unicursal

* American Journal of Mathematics, vol. XXIII, p. 1.

sein, werden also wegen des Complexes elliptischer Curven, den sie enthalten, ein rational-distinctes M_2-System vom Range 1 tragen, das dann aus allen simultan durch ein M_{r-3}-System ausgeschnitten werden muss. Indem wir aus den $r-3$ Systemen durch Festzetzung rational distincter Punkte Untersysteme auswählen, können wir wie in §11 auf jenes $\infty^3 M_{r-8}$-System eine birationale Transformation gründen, die es in ein ∞^3-System von R_{r-3} überführt und weiterhin die $r-3$ M_{r-1}-Systeme in R_{r-4}-Monoide (wie in §11).

THEOREM XII.—*Jeder uneigentliche Complex elliptischer Curven des R_r, dessen erzeugte M_2 nicht den Kegelflächentypus haben, ist in rational distincter Weise räumlich birational äquivalent einem Complexe von Curven, welche als Schnitt von $r-3$ Systemen von R_{r-4}-Monoiden und zwei Systemen von R_{r-4}-Kegeln erzeugt wird, welche sämmtlich denselben Scheitel-R_{r-4} haben. Die beiden Kegelsysteme bestimmen sich gemäss den beiden erzeugenden M_2-Systemen im R_3.*

Wir können endlich aus allen bisher erhaltenen Resultaten die wichtige Folgerung ziehen:

THEOREM XIII.—*Für jeden eigentlichen oder uneigentlichen linearen Complex elliptischer Curven im R_r, ausgenommen den eigentlichen ∞^{r-1}-Complex, lässt sich in rational distincter Weise ein lineares ∞^r-System Transversal-M_{r-1}, das den Rang 1 hat, angeben.*

§14.—*Die M_i im R_r, welche elliptische ebene Schnittcurven haben.*

Nachdem ich bereits im *American Journal*, vol. XXIII das Theorem von Picard über die Flächen M_2 mit rationalen ebenen Schnittcurven auf den R_r verallgemeinert habe, kann ich nun aus §§12 und 13 eine ähnliche Folgerung für die M_i mit elliptischen ebenen Schnitten ziehen.

Wie in §3 kann mit Hilfe derjenigen Schnittcurven, welche in den Tangentenräumen, also rational sind und welche überdies ihre Doppelpunkte längs einer solchen rationalen ebenen Curve, dann einer unicursalen M_2 etc. besitzen, bewiesen werden, dass die M_i abbildbar ist, sofern nicht die Tangenten-R_{r-1} ∞^1 lineare Systeme bilden, welche ∞^1-Reihe dann elliptisch sein muss.

Ist nun die M_i einmal unicursal, also abbildbar auf den R_i, dann muss das Abbildungssystem eines der in §§11, 12 kennen gelernten M_{i-1}-Systeme des R_i sein, das Bild der ebenen Schnittcurven muss einer jener typischen Complexe, sein, da eine birationale Umänderung des Abbildungssystemes gestattet ist. Diese M_{i-1}-Systeme belehren nun darüber, dass auf den M_i ein lineares System

von R_{i-3} bestehen muss, entsprechend den R_{i-3}, welche durch den Scheitel der R_{i-4}-Monoide der Abbildung gehen. Daher:

THEOREM XIV.—*Jede M_i im R_r, welche durchaus elliptische ebene Schnittcurven besitzt, erhält entweder $\infty^1 R_{i-1}$, welche eine elliptische Reihe bilden, oder $\infty^2 R_{i-2}$, oder $\infty^3 R_{i-3}$, welche je eine unicursale Reihe bilden, deren Beschaffenheit aus Theorem X. abzulesen ist.*

AD NOTAM.—In Ergänzung meiner chronologischen Feststellungen, die ich in der Fussnote zu pag. 1 gemacht habe, will ich hier angeben, welche Fortschritte meine Arbeit auch für den R_3, auf den F. Enriques sich überhaupt beschränkt, gegenüber der Enriques'schen Arbeit in Math. Ann., Bd. XLVI. leistet.

Enriques gesteht selbst ein, dass ihm die Behandlung des Problemes im R_3 mittelst einer directen Methode zu schwierig war und benützt dazu den R_4 gemäss einer von Riemann-Clifford-Veronese ausgebildeten Methode. Es sagt auf pag. 100: "Non potevo qui trarre grandi ajuti dalla teoria delle trasformazioni .... ; era dunque naturale che cercassi un ausilio nello studio projettivo delle varietà di 3 dimensioni a curve sezioni iperellittiche."

Nun *ich* erledige das Problem mittelst einer *directen* Methode, nämlich mittelst des von mir zuerst im *American Journal*, vol. XXIII ausgebildeten Transversalenprincipes. Wie ich schon daselbst erwähnt habe, ist der Character meiner Methode geradezu entgegengesetzt dem des italienischen Verfassers, insofern meine vom niederen zum höheren Raume fortschreitet, jener aber vom höheren in den niederen Raum herunter steigt.

Ein zweiter Fortschritt ist der, dass ich auch die ∞^r-Systeme, also im R_3 die ∞^3-Systeme zu erledigen im Stande bin. Ein dritter Fortschritt ist, dass ich die Typen nicht nur gegenüber der älteren Auffassung sondern auch gegenüber deren von mir zum ersten Male zur Geltung gebrachten Principe der durchgehenden Vermeidung alles Irrationalen aufstelle. Ein vierter Fortschritt, der meiner Methode inhärent ist, ist die in meiner Typentafel hervortretende ganz natürliche Eintheilung der eigentlichen Complexe in V Classen, während Enriques seine Typen im R_3 ganz pêle-mêle gibt, wie sie eben bei seiner Methode erscheinen.

Der bedeutendste (fünfte) Fortschritt aber ist der, dass Enriques überhaupt die von mir als uneigentlich bezeichneten Complexe nicht untersucht, während ich auch für sie die Typen angebe. Aber gerade die uneigentlichen Complexe sind für die Theorie der Transformationsgruppen die wichtigsten.

Auch erscheint es mir sechstens nothwendig, bei einigen Typen nicht bloss

anzugeben, dass die Basiscurven zerfallen können, sondern auch welche Zerfällungen—es sind dies sehr wenige—wieder auf neue Typen führen. Dies wird besonders dann beim Fortschreiten zum R_r wichtig.

Indem ich die Fussnote auf pag. 1. in meinem Interesse, die hier vorhergehende Aufzählung im Interesse der Sache gemacht habe, ist es nicht mein Wunsch, dem Verdienste des italienischen Verfassers, dessen leider sehr verstreute Arbeiten genauer Lectüre werth sind, um das R_3-Problem nahe zu treten. Aber eine Vergleichung meines §10 mit der Aufzählung in der Enriques'schen Abhandlung des Bandes XLVII der Math. Ann. wird zeigen, dass er ausserdem auch einige Fehler begangen hat, auch innerhalb der Beschränkung, die er sich nothwendiger Weise auferlegt. Ihm fehlen die Systeme n. 4. 5. 6 der Classe I. das System n. 4. der Classe III. Im Bande XLIX bezeichnet er dann die M_2^2, Bild aller C_3 der Ebene als einen Typus, was sie nicht ist, da das Gesammtsystem aller C_3 ein rational distinctes ∞^2-System von Transversalen gestattet. Ferner muss für die Anwendungen, wie meine Arbeit erst deutlich macht, jene "Nöthersche Doppelebene" durch die M_2^4 mit tacnode ersetzt werden, u. A.

INHALTSVERZEICHNIS.

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Generalization of the Differentiation Process.

By Robert E. Moritz.

1.—Notation and Definitions.

For the purpose of this discussion, we shall use the following notation:

The symbol) represents any process or operation whatsoever between any two concepts a and b, thus:

$$a\ )\ b = c,$$

where c is the result of the operation, a is the operand, b the operator of the process. If we look upon a as the operator and upon b as the operand, we write

$$b\ (\ a = c,$$

where the symbol (represents, of course, a new operation.

For every combinatory process), there exist, in general, two dissociating processes known as inverses. We denote these by the symbols $\smile$ and $\frown$, the former denoting the dissociation of c and b, the latter the dissociation of c and a, thus:

$$c \smile b = a, \qquad c \frown a = b.$$

and we refer to these processes as the first and second inverses of) respectively. Hence, they are likewise the second and first inverses of (respectively.

We may look upon c and b as the original concepts, and upon a as the result obtained from their combination, that is, we may look upon either inverse as the direct process and upon the former direct process as one of the inverses of the inverse.

If, among the process and its inverses, there is one which is single valued, that one is selected as the direct process. In this paper we shall assume throughout the singlevaluedness of the direct processes.

In special cases) and (may denote the same operation, and in such cases we shall represent the operation by |, a symbol devoid of aspect. It immediately follows that in this case $\smile$ and $\frown$ will also be the same operation which we will denote by —.

When relations are considered which subsist not only for the direct processes) and (, but for their inverses $\smile$ and $\frown$ as well, the symbol $\widetilde{)(}$ might be employed, this symbol being merely a combination of the others; in cases where the two inverses are identical, the symbol is contracted into $\top$. This latter form corresponds to the well-known symbols $\mp$, $\dot{\times}$ of the ordinary algebra. When we deal with a second operation, we represent it by)) and its inverses by $\succsim$ and $\precsim$ respectively. To the symbol | corresponds || and to $\top$ corresponds $\overline{\overline{\top}}$.*

We shall have occasion to deal with associated processes, processes which are related to each other by definite laws. The addition, multiplication, and involution processes of the ordinary algebra are such processes. We shall distinguish associated processes from each other by using as symbols for them) with various indices. Suppose that we have selected a reference process, say

$$a \,)_0\, b = c,$$

then we may write

$$\begin{aligned}
&a \,)_0\, a \,)_0 \,\ldots\, \text{to } \omega_0(b) \text{ terms} = a \,)_1\, b,\\
&a \,)_1\, a \,)_1 \,\ldots\, \text{“ } \omega_1(b) \quad \text{“} \quad = a \,)_2\, b,\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&a \,)_n\, a \,)_n \,\ldots\, \text{“ } \omega_n(b) \quad \text{“} \quad = a \,)_{n+1}\, b,
\end{aligned}$$

where $\omega_0(b), \omega_1(b), \ldots, \omega_n(b)$ represent arbitrarily selected functions of b. We shall refer to such processes as processes of the zero, first, second, ... n^{th}, $(n+1)^{\text{th}}$ orders respectively, the reference process being of the zero order.

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* I am aware that I am guilty of adding another set of symbols to the already too long list of symbols with a similar meaning used by others. Grassmann uses $\frown$ $\smile$ and also $\precsim$ $\succsim$ to denote direct and inverse combinations respectively, De Morgan uses λ γ, Hankel λ θ, Stolz $\bigcirc$ $\frown$ and also $\odot$ $\overset{\frown}{\cdot}$, Houel $\frown$ $\smile$ and $\uparrow\downarrow$. Davis suggests the use of any letter which has aspect as U, Y, or V; $a \,Y\, b$ or $b \,\lambda\, a$ could be used to denote any combination between a and b, $a \prec b$ and $a \succ b$ its two inverses, ψ ϕ, and φ ϕ have also been suggested, and there are no doubt many others which I have not seen. Where there are so many symbols to choose from, it is easier to devise a new symbol than to discriminate justly among the old.

When $\omega_0(b), \omega_1(b), \ldots \omega_n(b)$ become all equal to b, $)_0$ may be taken equal to $+$, and $)_1$, $)_2$ become the $\times$, $(\ )^{(\ )}$ of the ordinary algebra.

Any process may be selected as the reference process and negative indices may be used as well as positive. Thus we may have processes defined as follows:

$$\begin{array}{llll} a)_{-n} \quad a)_{-n} & \ldots \text{ to } \omega_{-n}(b) & \text{terms} = a)_{-n+1} b, \\ a)_{-n+1} a)_{-n+1} & \ldots \text{ “ } \omega_{-n+1}(b) & \text{“ } = a)_{-n+2} b, \\ \vdots \qquad \vdots & \vdots & \vdots \\ a)_{-1} \quad a)_{-1} & \ldots \text{ “ } \omega_{-1}(b) & \text{“ } = a)_0 b, \end{array}$$

which shows how the process $a)_0 b$ and hence any higher process may be defined in terms of any negative process. The first and second inverses of $)_n$ are, of course, $\smile_n$ and $\frown_n$ respectively.

The laws of combination, to which different operations are subject, cannot be determined *a priori*. They depend upon the nature of the concepts involved, the relations which subsist among the combinations employed, and upon the meanings attached to the results of the combinations. In a study of pure form, we may posit any laws of combination and examine the consequences following their assumption. In that case the concepts, among which the combinations take place, are in part defined by these laws, and hence may or may not have an existence in the world of experience. On the other hand, we may limit our discussion to such laws of combination and their consequences as are suggested by experience. There are five such laws which may be put under three general heads and are defined as follows:

1.—*The Commutative Law*, $a)b = b)a$. This law is characteristic of any process denoted by $|$ or $\|$.

2.—*The Associative Law*, $a)_n b)_n c = a)_n (b)_m c)$, or when it is expressed in the most general form possible

$$f(a, b, c) = \phi[a, \psi(b, c)].$$

3.—*The Distributive Law*, the common form of which comes under the form $a)_n (b)_p c) = a)_n b)_q (a)_n c)$, and this is but a special case of the still more general form

$$F[a, f(b, c)] = \Phi[\phi(a, b), \psi(a, c)].$$

Among the symbols which enter into combination with each other, there is frequently one, M_n, which satisfies the equation $a)_n M_n = a$, no matter what a is. Such a symbol is called the *modulus* of the process under consideration. The

second inverse gives immediately $a \frown_n a = M_n$. Also $a \smile_n M_n = a$, that is, M_n is likewise a modulus of the first inverse.

2.—PRELIMINARY THEOREMS.

Grassmann,* Hankel,† and Stolz‡ have considered at length single associative processes both with and without commutation. Interesting results have been reached by them, as for instance that the formulæ

$$a \smile (b) c) = a \smile b \smile c \quad \text{and} \quad a \smile (b \smile c) = a \smile b) c$$

involve the commutative law, and that a combinatory process which is commutative for two terms and associative for three, is commutative and associative for any number of terms. Equally interesting is the study of the theory of associated processes. We shall prove three theorems.

THEOREM I.—*If an associated process of the n^{th} order admits a modulus M_n, then $M_n \frown_{n+1} M_n$ is indeterminate.*

We have, by definition and hypothesis,

$$a)_n M_n = a,$$

and, therefore,

$$a)_n M_n)_n M_n)_n \ldots \text{ to } f(b) \text{ terms} = a.$$

a may be taken equal to M_n, hence,

$$M_n)_n M_n)_n \ldots \text{ to } f(b) \text{ terms} = M_n,$$

that is,

$$M_n)_{n+1} b = M_n.$$

The second inverse of this last equation yields

$$M_n \frown_{n+1} M_n = b, \text{ no matter what } b \text{ is.}$$

Familiar illustrations of this theorem from the ordinary algebra are the forms $\frac{0}{0}$ and $\log_1 1$.

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* Grassmann, "Ausdehnungslehre," 1844.

† Hankel, "Vorlesungen über Complexe Zahlen."

‡ Stolz, "Algemeine Arithmetik," Bd. 1.

THEOREM II.—*If two associated processes are subject to an associative law, then*

$$p \frown_n q = p \frown_n a \frown_m (q \frown_n a);$$

if, moreover, one of the processes admits a modulus, then

$$p \frown_n q = M_n \frown_m (q \frown_n p).$$

The proof is simple. Let

$$a)_n b)_n c = p,$$

and $$a)_n b = q,$$

then will $$q)_n c = p.$$

By hypothesis $$a)_n b)_n c = a)_n (b)_m c).$$

Let $$b)_m c = r,$$

then $$a)_n r = p.$$

The second inverses of these processes are

$$q \frown_n a = b,$$
$$p \frown_n q = c,$$
$$r \frown_m b = c,$$
$$p \frown_n a = r.$$

Now substitute in the third of the last four equations, the values of r, b and c, from the other three, and we get

$$p \frown_n a \frown_m (q \frown_n a) = p \frown_n q.$$

If, now, the n^{th} process admits a modulus, we have, by putting p for a in the last equation,

$$p \frown_n p \frown_m (q \frown_n p) = p \frown_n q,$$

or $$M_n \frown_m (q \frown_n p) = p \frown_n q.$$

If, on the other hand, the m^{th} process has a modulus, we have, by putting p for q

$$p \frown_n a \frown_m (p \frown_n a) = p \frown_n p,$$

or $$M_m = p \frown_n p.$$

This last equation holds for any p, hence the n^{th} process has a modulus, and the theorem follows as before.

In the ordinary algebra, the m in our statement of the associative principle

is either n or $n-1$, and hence our theorem gives the forms,

for $m=n$,	for $m=n-1$,
$p \frown_n q = p \frown_n a \frown_n (q \frown_n a)$,	$p \frown_n q = p \frown_n a \frown_{n-1} (q \frown_n a)$,
$p \frown_n q = \quad M_n \frown_n (q \frown_n p)$,	$p \frown_n q = \quad M_n \frown_{n-1} (q \frown_n p)$.

For $n=0$, 1 and 2 successively, we get the familiar equations

$$p-q=p-a-(q-a), \qquad \log_q p = \frac{\log_a p}{\log_a q};$$
$$p \div q = p \div a \div (q \div a);$$

and

$$p-q=-(q-p), \qquad \log_q p = \frac{1}{\log_p q}.$$
$$p \div q = 1 \div (q \div p).$$

THEOREM III.—*If three associated processes, of the* $(n-1)^{\text{th}}$, n^{th} *and* m^{th} *orders respectively, of which the first and third admit each a modulus,* M_{n-1} *and* M_m, *are subject to a distributive law, then* $M_{n-1} \frown_n a = M_m$, *a being different from* M_{n-1}.

By hypothesis,

$$a)_n (b)_m c) = a)_n b)_{n-1} (a)_n c),$$

and, therefore,

$$a)_n c = a)_n (b)_m c) \frown_{n-1} (a)_n b).$$

Now put $b)_m c = b'$ and $c = b' \frown_m b$, and the last equation becomes

$$a)_n (b' \frown_m b) = a)_n b' \frown_{n-1} (a)_n b).$$

Let $b'=b$, $b' \frown_m b$ becomes M_m and the right member of the last equation becomes M_{n-1}, so that

$$a)_n M_m = M_{n-1}, \text{ unless } a = M_{n-1},$$

or, finally,

$$M_{n-1} \frown_n a = M_m.$$

If $a = M_{n-1}$, the left member of the last equation becomes $M_{n-1} \frown_n M_{n-1}$, which by theorem I, is indeterminate.

In the ordinary algebra, the distributive law holds only for $m=n-1$ and $m=n-2$. We obtain then the two forms

$$M_{n-1} \frown_n a = M_{n-1}, \quad a \neq M_{n-1},$$

and

$$M_{n-1} \frown_n a = M_{n-2}, \quad a \neq M_{n-1},$$

which, for $n=1$ and $n=2$ successively, reduce to

$$\frac{0}{a} = 0, \quad a \neq 0; \quad \text{and} \quad \log_1^a = 0, \quad a \neq 1.$$

3.—LIMITING PROCESSES ALLIED TO DIFFERENTIATION.

Using the general notation defined in art. 1 instead of that ordinarily employed in writing the differential coefficient of a function, the definition equation of the differential coefficient may be written as follows,

$$\frac{dy}{dx} = \lim_{h=M_0} \{F(x)_0 h) \frown_0 F(x) \frown_1 h\},$$

where $y = F(x)$.

This form of the differential coefficient suggests at once other processes which lead to the symbols

$$\frac{d_1 y}{d_1 x}, \quad \frac{d_2 y}{d_2 x}, \quad \ldots, \quad \frac{d_n y}{d_n x},$$

as defined by the following equations,

$$\begin{aligned}
\frac{d_1 y}{d_1 x} &= \lim_{h=M_1} \{F(x)_1 h) \frown_1 F(x) \frown_2 h\}, \\
\frac{d_2 y}{d_2 x} &= \lim_{h=M_2} \{F(x)_2 h) \frown_2 F(x) \frown_3 h\}, \\
&\vdots \qquad\qquad \vdots \\
\frac{d_n y}{d_n x} &= \lim_{h=M_n} \{F(x)_n h) \frown_n F(x) \frown_{n+1} h\}.
\end{aligned}$$

The ordinary differential coefficient is the $\frac{d_0 y}{d_0 x}$ belonging to this chain of expressions.

Consider the process defined by the last equation. We pass to the limit indicated and have

$$\begin{aligned}
\frac{d_n y}{d_n x} &= \lim_{h=M_n} \{F(x)_n h) \frown_n F(x) \frown_{n+1} h\} \\
&= F(x)_n M_n) \frown_n F(x) \frown_{n+1} M_n \\
&= F(x) \frown_n F(x) \frown_{n+1} M_n = M_n \frown_{n+1} M_n.
\end{aligned}$$

The right member of the last equation is indeterminate by theorem I of the preceding section, and gives rise to an algebraic limiting process, provided that a process of the n^{th} order exists, and that this process admits a modulus. The evaluation of $\frac{d_n y}{d_n x}$ may be considered, and if it can be effected, will give rise to a set of forms of which the ordinary differential coefficients constitute a particular

set. In some cases the evaluation can be readily effected. If, for instance, an associative and distributive law obtains, as in the case of the involution process of the ordinary algebra, then

$$\frac{d_n y}{d_n x} = M_n \frown_{n+1} M_n = M_n \frown_{n+1} a \frown_n (M_n \frown_{n+1} a)$$

by aid of theorem II of the last section, and this in turn, by applying theorem III, may be transformed into

$$\frac{d_n y}{d_n x} = M_{n-1} \frown_n M_{n-1},$$

that is, $M_n \frown_{n+1} M_n$ can be evaluated, provided $M_{n-1} \frown_n M_{n-1}$ can be. In this case, and others that will be treated later, $\frac{d_n y}{d_n x}$ can be expressed as a function of $\frac{d_{n-1} y}{d_{n-1} x}$.

4.—QUOTIENTIAL COEFFICIENTS.

Let us now consider more closely the process defined by

$$\frac{d_1 y}{d_1 x} = \lim_{h = M_1} \{ F(x)_1 h) \frown_1 F(x) \frown_2 h \}.$$

When the operations involved in this equation are those of the ordinary algebra, we shall write $\frac{qy}{qx}$ for $\frac{d_1 y}{d_1 x}$, so that

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \log_h \frac{F(xh)}{F(x)} \right\},$$

from which follows

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \frac{\log \frac{F(xh)}{F(x)}}{\log h} \right\}.$$

This is of the form $\frac{0}{0}$ and may hence be evaluated by well known rules. We have

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \frac{h F'_h(xh)}{F(xh)} \right\}, \text{ where } F'_h(xh) = \frac{dF(xh)}{dh},$$

but

$$\lim_{h=1} \left\{ F'_h(xh) \right\} = x \frac{dFx}{dx} = xF'(x),$$

so that $$\frac{qy}{qx} = \frac{xF'(x)}{F(x)} = \frac{x}{y} \cdot \frac{dy}{dx} = \frac{d \log y}{d \log x}.$$

Or we proceed as follows

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \log_h \frac{f(xh)}{f(x)} \right\} = \lim_{h=1} \left\{ \frac{\log f(xh) - \log f(x)}{\log h} \right\}$$

$$= \lim_{h'=0} \left\{ \frac{F(z+h') - F(z)}{h'} \right\},$$

where $h' = \log h$, $z = \log x$, $F(z) = \log f(x)$, hence,

$$\frac{qy}{qx} = \frac{d F(z)}{dz} = \frac{d \log f(x)}{d \log x} = \frac{d \log y}{d \log x} = \frac{x}{y} \cdot \frac{dy}{dx}.$$

We call the form thus derived the *quotiential coefficient* of the function $y = f(x)$, and refer to the process of deriving it as *quotientiation*. The following table gives a partial list of quotiential coefficients:

(1). $\frac{q(a)}{qx} = 0$, a being any constant.

(2). $\frac{q(x)}{qx} = 1.$ (3). $\frac{q(ax^n)}{qx} = n.$

(4). $\frac{q(ax^n \pm bx^m)}{qx} = \frac{nax^n \pm mbx^m}{ax^n \pm bx^m}.$

(5). $\frac{q(\sin x)}{qx} = x \cot x.$ (6). $\frac{q(a^{\phi(x)})}{qx} = x \log a \cdot \phi'(x).$

(7). $\frac{q(e^x)}{qx} = x.$ (8). $\frac{q(\log x)}{qx} = \frac{1}{\log x}.$

(9). $$\frac{q(u+v+w+\ldots.)}{qx} = \frac{u\frac{qu}{qx} + v\frac{qv}{qx} + w\frac{qw}{qx} + \ldots.}{u+v+w+\ldots.},$$

$u, v, w, \ldots.$ being each a function of x.

(10). $\frac{q(u.v.w\ldots.)}{qx} = \frac{qu}{qx} + \frac{qv}{qx} + \frac{qw}{qx} + \ldots.$

(11). $\frac{q\left(\frac{u}{v}\right)}{qx} = \frac{qu}{qx} - \frac{qv}{qx}.$ (12). $\frac{q\left(\frac{1}{u}\right)}{qx} = -\frac{qu}{qx}.$

(13). $\frac{q(au)}{qx} = \frac{qu}{qx}.$ (14). $\frac{q(u^n)}{qx} = n\frac{qu}{qx}.$

(15). $\frac{q(u^v)}{qx} = \left(\frac{q(u^v)}{qu}\right)\frac{qu}{qx} + \left(\frac{q(u^v)}{qv}\right)\frac{qv}{qx} = v\left(\frac{qu}{qx} + \log u \frac{qv}{qx}\right),$

where $\left(\frac{q(u^v)}{qu}\right)$ signifies that v is to be considered constant, while the function is quotientiated with respect to u.

$$(16). \quad \frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx},$$

in which u is a function of z, and z in turn a function of x.

$$(17). \quad \frac{qu}{qx} = \frac{1}{\frac{qx}{qu}}.$$

5.—QUOTIENTIATION NOT DEPENDENT ON DIFFERENTIATION.

While, for the sake of convenience, we have made use of the well-known rules for differentiation in deriving the above formulæ, it is not necessary to do so, but each quotientiation formula can be derived by an independent limiting process. In fact, since

$$\frac{qy}{qx} = \frac{x}{y} \cdot \frac{dy}{dx},$$

each of the processes of differentiation and quotientiation can be expressed in terms of the other. Having once established independently the rules for quotientiation, all differentiation formulæ may be deduced from them. Some differentiation formulæ are thus more easily derived than by the ordinary methods. We proceed to derive a few of the leading quotientiation formulæ without the aid of differentiation.

(a).—*Quotiential Coefficient of* ax^n.

By definition

$$\frac{q(ax^n)}{qx} = \lim_{h=1} \left\{ \log_h \left[\frac{ax^n h^n}{ax^n} \right] \right\} = \lim_{h=1} \left\{ \log_h h^n \right\} = \lim_{h=1} (n) = n.$$

(b). *Quotiential Coefficient of* e^x.

By definition

$$\frac{q(e^x)}{qx} = \lim_{h=1} \left\{ \log_h \left(\frac{e^{xh}}{e^x} \right) \right\} = \lim_{h=1} \left\{ \frac{xh - x}{\log h} \right\} = x \cdot \lim_{h=1} \left\{ \frac{h-1}{\log h} \right\}.$$

Put $h-1=h'$, then

$$\frac{h-1}{\log h}=\frac{h'}{\log(1+h')}=\frac{h'}{h'-\frac{h'^2}{2}+\frac{h'^3}{3}-\text{etc.}}$$

and
$$\lim_{h=1}\left\{\frac{h-1}{\log h}\right\}=\lim_{h'=0}\left\{\frac{1}{1-\frac{h'}{2}+\frac{h'^2}{3}-\text{etc.}}\right\}=1.*$$

Therefore, $\frac{q(e^x)}{qx}=x.$

(c). *Quotientiation of the Trigonometric Functions.*

$$\frac{q(\sin x)}{qx}=\lim_{h=1}\left\{\log_h\left(\frac{\sin xh}{\sin x}\right)\right\}=\lim_{h=1}\left\{\frac{\log\left(\frac{\sin xh}{\sin x}\right)}{\log h}\right\}.$$

Put $h=1+m$, then

$$\lim_{h=1}\left\{\frac{\log\left(\frac{\sin xh}{\sin x}\right)}{\log h}\right\}=\lim_{m=0}\left\{\frac{\log\frac{\sin x\cos mx+\cos x\sin mx}{\sin x}}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{\log(\cos mx+\cot x\sin mx)}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{\log(1+mx\cot x)}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{mx\cot x-\frac{(mx\cot x)^2}{2}+\text{etc.}}{m-\frac{m^2}{2}+\text{etc.}}\right\}=x\cot x,$$

that is,
$$\frac{q(\sin x)}{qx}=x\cot x.$$

(d). *Quotiential Coefficient of a Function of a Function.*

Let $u=f(z)$ and $z=\phi(x)$, and suppose that the substitution of $\phi(x)$ for z in $f(z)$ gives $u=F(x)$. Let us change x into xh, in consequence of which z becomes zk and u becomes ul, thus, $zk=\phi(xh)$, $ul=f(zk)$; also $ul=F(xh)$,

* The development of $\log(1+h)$, and its convergence is readily established without the aid of the differential calculus.

since x and u have obviously the same values in the third equation as in the first two.

Now, by definition, we have

$$\frac{qu}{qz} = \lim_{k=1}\left\{\log_k\left[\frac{f(zk)}{f(z)}\right]\right\} = \lim_{k=1}\left\{\log_k\left(\frac{ul}{u}\right)\right\} = \lim_{k=1}(\log_k l),$$
$$\frac{qz}{qx} = \lim_{h=1}\left\{\log_h\left[\frac{\phi(xh)}{\phi(x)}\right]\right\} = \lim_{h=1}\left\{\log_h\left(\frac{zk}{z}\right)\right\} = \lim_{h=1}(\log_h k).$$

Multiplying these two equations, remembering that k, h, and l approach unity together, we have

$$\frac{qu}{qz}\cdot\frac{qz}{qx} = \lim_{h=1}\{\log_k l\,.\log_h k\} = \lim_{h=1}\left\{\frac{\log l}{\log k}\cdot\frac{\log k}{\log h}\right\} = \lim_{h=1}\{\log_h l\}$$
$$= \lim_{h=1}\left\{\log_h\left(\frac{ul}{u}\right)\right\} = \lim_{h=1}\left\{\log_h\left[\frac{F(xh)}{F(x)}\right]\right\} = \frac{qu}{qx}.$$

6.—PARALLELISMS BETWEEN QUOTIENTIAL AND DIFFERENTIAL PROCESSES.

We propose now to trace some of the more important parallelisms which exist between quotiential and differential processes. In most cases, a mere inspection of the formulæ in §4 will justify the theorems as announced, in others what proof is necessary will be added. We shall use the abbreviations q. c. for quotiential coefficient and d. c. for differential coefficient.

(a). *Both processes are distributive, the one over a product, the other over a sum of functions,*

$$\frac{q(u_1\,.\,u_2\,.\,\ldots\,.\,u_n)}{qx} = \sum_{i=1}^{n}\frac{qu_i}{qx}, \qquad \frac{d(u_1+u_2+\ldots+u_n)}{dx} = \sum_{i=1}^{n}\frac{du_i}{dx},$$

$u_1, u_2, \ldots, u_n$ being each a function of x.

If $(u_i = u_k)_{i,\,k,\,=1,\,2,\,\ldots\,n}$, there follow immediately the corollaries:

The q. c. of a function powered by a constant, is equal to that function multiplied by the q. c. of the variable,

$$\frac{q(u^n)}{qx} = n\frac{qu}{qx},$$
$$\frac{q(x^n)}{qx} = n.$$

The d. c. of a function multiplied by a constant, is equal to that constant multiplied by the d. c. of the function,

$$\frac{d(nu)}{dx} = n\frac{du}{dx},$$
$$\frac{d(nx)}{dx} = n.$$

(b). *The result of the quotientiation is not changed if the function is multiplied or divided by a constant,*

$$\frac{q(u \dot{\times} a)}{qx} = \frac{qu}{qx}.$$

The result of differentiation is not changed if the function is increased or diminished by a constant,

$$\frac{d(u \pm a)}{dx} = \frac{du}{dx}.$$

(c). *The q. c. of the reciprocal of a function is equal to the q. c. of the function with its sign changed,*

$$\frac{q\left(\frac{1}{u}\right)}{qx} = -\frac{qu}{qx}.$$

The d. c. of the negative of a function is equal to the d. c. of the function with its sign changed,

$$\frac{d(-u)}{dx} = -\frac{du}{dx}.$$

(d).

$$\frac{q(u_1 + u_2 + \ldots + u_n)}{qx} = \frac{1}{F}\sum_{i=1}^{n} u_i \frac{qu_i}{qx},$$

$$\frac{d(u_1 . u_2 \ldots . u_n)}{dx} = \sum_{i=1}^{n} u_i \frac{du_i}{dx}.$$

where $F = u_1 + u_2 + \ldots + u_n$.

(e). $y = \log x$, when quotientiated becomes its reciprocal, $y = \frac{-1}{\log x}$ remains unchanged.

$y = e^x$, when differentiated, remains unchanged, $y = \sqrt{2x}$, becomes its reciprocal.

(f). *The q. c. of a function of a product of two independent variables is the same, whether we quotientiate with respect to one or the other of the variables,*

$$\frac{qf(xy)}{qx} = \frac{qf(xy)}{qy}.$$

The d. c. of a function of the sum of two independent variables is the same whether we differentiate with respect to one or the other of the variables,

$$\frac{df(x+y)}{dx} = \frac{df(x+y)}{dy}.$$

This theorem may be proven as follows:

$$\frac{qf(xy)}{qx} = \lim_{h=1}\left\{\log_h \frac{f(xh . y)}{f(xy)}\right\}$$

$$= \lim_{h=1}\left\{\log_h \frac{f(x . yh)}{f(xy)}\right\} = \frac{qf(xy)}{qy}.$$

(g). *If* $u = F(z)$, $z = f(x)$, *we have*

$$\frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx}. \qquad \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}.$$

(h). *If* u *and* v *are both functions of* x, *we have*

$$\frac{q(u^v)}{qx} = \left(\frac{q(u^v)}{qu}\right)\frac{qu}{qx} + \left(\frac{q(u^v)}{qv}\right)\frac{qv}{qx}, \qquad \frac{d(u^v)}{dx} = \frac{\partial(u^v)}{\partial u} \cdot \frac{du}{dx} + \frac{\partial(u^v)}{\partial v} \cdot \frac{dv}{dx},$$

and, generally, if F *is a function of* $u_1, u_2, \ldots, u_n$, *where* $u_1, u_2, \ldots, u_n$, *are each functions of* x, *we have*

$$\frac{qF}{qx} = \sum_{i=1}^{n}\left(\frac{qF}{qu_i}\right)\frac{qu_i}{qx}. \qquad \frac{dF}{dx} = \sum_{i=1}^{n}\frac{\partial F}{\partial u_i} \cdot \frac{du_i}{dx}.$$

(i). *Implicit functions.*

If in the last equation $F = 0$, it can readily be shown that

$$\left(\frac{qF}{qu_1}\right)\frac{qu_1}{qx} + \left(\frac{qF}{qu_2}\right)\frac{qu_2}{qx} + \ldots\ldots\ldots\ldots\ldots\left(\frac{qF}{qu_n}\right)\frac{qu_n}{qx} = 0.$$

For $F = F(x, y) = 0$, the equation becomes $\left(\frac{qF}{qx}\right) + \left(\frac{qF}{qy}\right)\frac{qy}{qx} = 0$, and hence

$$\frac{qy}{qx} = -\frac{\left(\frac{qF}{qx}\right)}{\left(\frac{qF}{qy}\right)}.$$

(j). *Successive Quotientiation.*

In general, the q. c. of a function of x will be again a function of x, and the q. c. of this function may be found by the preceding rules. We may, following the nomenclature of the differential calculus, designate this q. c. as the second quotiential coefficient of the original function, and write

$$\frac{q^2y}{qx^2} \text{ for } \frac{q\left(\frac{qy}{qx}\right)}{qx},$$

and generally $\quad \frac{q^ny}{qx^n}$ for $\left(\frac{q}{qx}\right)^n y,$

just as we write $\quad \frac{d^ny}{dx^n}$ for $\left(\frac{d}{dx}\right)^n y.$

(k). *Change of independent variable.*

Since
$$\frac{qy}{qx} = \frac{1}{\dfrac{qx}{qy}}, \quad \frac{qy}{qx} = \frac{qy}{qz} \cdot \frac{qz}{qx} = \frac{qy}{qz} \Big/ \frac{qx}{qz},$$

and hence

$$\frac{q^2y}{qx^2} = \left(\frac{q^2y}{qz^2} - \frac{q^2x}{qz^2}\right) \Big/ \frac{qx}{qz},$$

$$\frac{q^3y}{qx^3} = \left[\frac{\dfrac{q^2y}{qz^2} \cdot \dfrac{q^3y}{qz^3} - \dfrac{q^2x}{qz^2} \cdot \dfrac{q^3x}{qz^3}}{\dfrac{q^2y}{qz^2} - \dfrac{q^2x}{qz^2}} - \frac{q^2x}{qz^2}\right] \Big/ \frac{qx}{qz},$$

etc. etc. formulæ which must be used, when we wish to change the independent variable from x to z. When $z = y$, these formulæ become

$$\frac{qy}{qx} = 1 \Big/ \frac{qx}{qy}, \quad \frac{q^2y}{qx^2} = -\frac{q^2x}{qy^2} \Big/ \frac{qx}{qy},$$
$$\frac{q^3y}{qx^3} = \left(\frac{q^3x}{qy^3} - \frac{q^2x}{qy^2}\right) \Big/ \frac{qx}{qy}, \quad \text{etc. etc.}$$

7.—FUNCTIONS OF TWO INDEPENDENT VARIABLES.

THEOREM I.—*If $u = f(x + y)$, where x and y are independent variables, then*

$$\frac{qu}{qx} : \frac{qu}{qy} = x : y.$$

For, if we put $x + y = z$, we have

$$\frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx}, \quad \text{and} \quad \frac{qu}{qy} = \frac{qu}{qz} \cdot \frac{qz}{qy};$$

but
$$\frac{qz}{qx} = \frac{x}{x+y}, \quad \text{and} \quad \frac{qz}{qy} = \frac{y}{x+y}.$$

Hence
$$\frac{qu}{qx} : \frac{qu}{qy} = \frac{qu}{qz} \cdot \frac{x}{x+y} : \frac{qu}{qz} \cdot \frac{y}{x+y} = x : y.$$

THEOREM II.—*If $u = f(xy)$, where x and y are independent variables, then*

$$\frac{q^n u}{qx^n} = \frac{q^n u}{qy^n}.$$

Let us assume that the theorem holds for some index r, we will show that it then holds for the next higher index $r+1$

Put $$\frac{q^r u}{qx^r} = \frac{q^r u}{qy^r} = v, \quad \text{and} \quad xy = z,$$

then $$\frac{q^{r+1}u}{qx^{r+1}} = \frac{qv}{qx} = \frac{qv}{qz}\cdot\frac{qz}{qx},$$

and $$\frac{q^{r+1}u}{qy^{r+1}} = \frac{qv}{qy} = \frac{qv}{qz}\cdot\frac{qz}{qy},$$

but $$\frac{qz}{qx} = 1, \quad \text{and} \quad \frac{qz}{qy} = 1,$$

hence $$\frac{q^{r+1}u}{qx^{r+1}} = \frac{q^{r+1}u}{qy^{r+1}}.$$

Now the theorem has already been proven when $r=1$ (*f.* §6.), hence it holds when $r = 2, 3, 4$, or any number.

THEOREM III.—*If* $u = f(x^y)$, x *and* y *being independent, then*

$$\frac{qu}{qx} : \frac{qu}{qy} = 1 : \log x.$$

Put $z = x^y$, then $\frac{qz}{qx} = y$, and $\frac{qz}{qy} = y \log x$;

also

$$\frac{qu}{qx} = \frac{qu}{qz}\cdot\frac{qz}{qx} = y\cdot\frac{qu}{qz}, \quad \frac{qu}{qy} = \frac{qu}{qz}\cdot\frac{qz}{qy} = y\log x\,\frac{qu}{qz}.$$

Hence,

$$\frac{qu}{qx} : \frac{qu}{qy} = y\,\frac{qu}{qz} : y\log x\,\frac{qu}{qz} = 1 : \log x.$$

If we put $y = \log y'$, that is, if $u = f(x^{\log y'})$, the theorem assumes the symmetric form $\frac{qu}{qx} : \frac{qu}{qy'} = \log y' : \log x$.

THEOREM IV.—*If* $u = f(x, y)$, x *and* y *being independent, then*

$$\frac{q^2u}{qxqy}\cdot\frac{qu}{qy} = \frac{q^2u}{qyqx}\cdot\frac{qu}{qx}.$$

Proof:

$$\frac{q^2u}{qxqy} = \frac{q\left(\frac{y}{u}\cdot\frac{du}{dy}\right)}{qx} = \frac{x\left(\frac{d^2u}{dxdy} - \frac{1}{u}\cdot\frac{du}{dx}\cdot\frac{du}{dy}\right)}{\frac{du}{dy}},$$

$$\frac{q^2u}{qyqx} = \frac{q\left(\frac{x}{u}\cdot\frac{du}{dx}\right)}{qy} = \frac{y\left(\frac{d^2u}{dydx} - \frac{1}{u}\cdot\frac{du}{dy}\cdot\frac{du}{dx}\right)}{\frac{du}{dx}},$$

whence $$\frac{q^2u}{qxqy} : \frac{q^2u}{qyqx} = \frac{x}{\frac{du}{dy}} : \frac{y}{\frac{du}{dx}} = \frac{x}{u} \cdot \frac{du}{dx} : \frac{y}{u} \cdot \frac{du}{dy} = \frac{qu}{qx} : \frac{qu}{qy},$$

or $$\frac{q^2u}{qxqy} \cdot \frac{qu}{qy} = \frac{q^2u}{qyqx} \cdot \frac{qu}{qx}.$$

Cor. 1. *If* $u = f(x+y)$, *then* $\frac{q^2u}{qx\,qy} \cdot y = \frac{q^2u}{qy\,qx} \cdot x$, for in that case it has been shown in theorem I, $\frac{qu}{qx} : \frac{qu}{qy} = x : y$.

Cor. 2. In order that $\frac{q^2u}{qx\,qy} = \frac{q^2u}{qy\,qx}$, we must have $\frac{qu}{qx} = \frac{qu}{qy}$, hence, *the necessary and sufficient condition that the same result be obtained whether a function be quotientiated first with respect to* x *and then with respect to* y, *or first with respect to* y *and then with respect to* x, *is that* u *satisfies the relation* $\frac{qu}{qx} = \frac{qu}{qy}$. $u = f(xy)$ evidently satisfies this criterion.

8.—FUNCTIONS OF THREE OR MORE INDEPENDENT VARIABLES.

Let, now, $u = f(x, y, z)$, where x, y and z are independent variables, we then have, by theorem IV of the last paragraph,

$$\frac{q^2u}{qy\,qz} = \frac{q^2u}{qz\,qy} \cdot \frac{qu}{qy} \Big/ \frac{qu}{qz},$$

and hence

$$\frac{q^3u}{qx\,qy\,qz} = \frac{q\left(\frac{q^2u}{qy\,qz}\right)}{qx} = \frac{q^3u}{qx\,qz\,qy} + \frac{q^2u}{qx\,qy} - \frac{q^2u}{qx\,qz},$$

or $$\frac{q^3u}{qx\,qy\,qz} + \frac{q^2u}{qx\,qz} = \frac{q^3u}{qx\,qz\,qy} + \frac{q^2u}{qx\,qy}. \qquad [A]$$

Again, putting $v = \frac{qu}{qz}$, we have

$$\frac{q^3u}{qx\,qy\,qz} = \frac{q^2v}{qx\,qy} = \frac{q^2v}{qy\,qx} \cdot \frac{qv}{qx} \Big/ \frac{qv}{qy} = \frac{q^3u}{qy\,qx\,qz} \cdot \frac{q^2u}{qx\,qz} \Big/ \frac{q^2u}{qy\,qz},$$

or $$\frac{q^3u}{qx\,qy\,qz} \cdot \frac{q^2u}{qy\,qz} = \frac{q^3u}{qy\,qx\,qz} \cdot \frac{q^2u}{qx\,qz}. \qquad [B]$$

Finally, multiply [A] by $\frac{q^2u}{qzqy}$,

$$\frac{q^3u}{qxqyqz}\cdot\frac{q^2x}{qzqy}+\frac{q^2u}{qxqz}\cdot\frac{q^2u}{qzqy}=\frac{q^3u}{qxqzqy}\cdot\frac{q^2u}{qzqy}+\frac{q^2u}{qxqy}\cdot\frac{q^2u}{qzqy},$$

from [B],

$$\frac{q^3u}{qxqzqy}\cdot\frac{q^2u}{qzqy}=\frac{q^3u}{qzqxqy}\cdot\frac{q^2u}{qxqy},$$

and interchanging x and z in [A] and multiplying by $\frac{q^2u}{qxqy}$,

$$\frac{q^3u}{qzqxqy}\cdot\frac{q^2u}{qxqy}+\frac{q^2u}{qzqy}\cdot\frac{q^2u}{qxqy}=\frac{q^3u}{qzqyqx}\cdot\frac{q^2u}{qxqy}+\frac{q^2u}{qzqx}\cdot\frac{q^2u}{qxqy}.$$

Adding the last three equations and simplifying the result we obtain

$$\Big[\frac{q^3u}{qxqyqz}+\frac{q^2u}{qxqz}+\frac{q^2u}{qxqy}\Big]\frac{q^2u}{qzqy}=\Big[\frac{q^3u}{qzqyqx}+\frac{q^2u}{qzqx}+\frac{q^2u}{qzqy}\Big]\frac{q^2u}{qxqy}. \qquad [C]$$

Formulæ [A], [B], [C], give the rules according to which we may interchange in any order whatsoever the quotientiation with respect to the three independent variables x, y, z. The law of the interchange of x and y in $\frac{q^3u}{qxqy^2}$ is obtained from [C] by putting $z=y$. We get

$$\Big[\frac{q^3u}{qxqy^2}+\frac{q^2u}{qxqy}\Big]\frac{q^2u}{qy^2}=\Big[\frac{q^3u}{qy^2qx}+\frac{q^2u}{qyqx}\Big]\frac{q^2u}{qxqy}.$$

We next consider the general problem: *Given a function of n variables $u=f(x_1, x_2, \ldots, x_n)$; required the rule by which the n^{th} q.c. of u with respect to the n variables $x_1, x_2, \ldots, x_n$, may be replaced by the n^{th} q.c. of u with respect to the same variables in the same order except that the quotientiation with respect to any two particular variables as x_i and x_k is interchanged.*

We shall find it convenient to use an abridged notation so we shall write

Q_xu for $\frac{qu}{qx}$, q_x^2u for $\frac{q^2u}{qx^2}$, similarly $Q^2_{x_1x_2}u\equiv\frac{q^2u}{qx_1qx_2}$, $Q^3_{x_1x_2x_3}u\equiv\frac{q^3u}{qx_1qx_2qx_3}$, etc.;

and finally, when there is no danger of confusion, we put Qu for Q_xu, Q^2u for Q^2_xu, $Q_{12}u$ for $Q^2_{x_1x_2}u$, $Q_{123}u$ for $Q^3_{x_1x_2x_3}u$, etc. Using this notation, the formulæ of theorem IV, §7. and [A], [B], [C] of this paragraph, become respectively

$$Q_{12}u \,.\, Q_2u = Q_{21}u \,.\, Q_1u,$$
$$Q_{123}u + Q_{13}u = Q_{132}u + Q_{12}u,$$
$$Q_{123}u \,.\, Q_{23}u = Q_{213}u \,.\, Q_{13}u,$$
$$[Q_{123}u + Q_{13}u + Q_{12}u]\, Q_{32}\, u = [Q_{321}u + Q_{31}u + Q_{32}u]\, Q_{12}u.$$

Let us now suppose that the laws for the interchange of variables in the q. c.'s of the n^{th} and lower orders are known, and that the equations expressing these laws are rational and linear in the q. c. of the highest order. Let $Q^{n+1}_{1,2,\ldots,i,\ \ ,k,\ldots,n+1}u$ be the q. c. in question, and let it be required to find the law for the interchange of x_i and x_k. Then three cases arise :

Case I.— When $i, k \neq 1$.

Under this assumption, we put

$$Q^{n+1}_{1,2,\ldots,i,\ldots,k,\ldots,n+1}u = Q_1 Q^{n}_{2,\ldots,i,\ldots,k,\ldots,n+1}u.$$

Now, the law for the interchange of variables in $Q^{n}_{2,\ldots,i,\ldots,k,\ldots,n+1}u$ is known, hence, if we quotientiate with respect to x the equation which expresses this law, we get an equation which is seen to involve only terms of the first degree in $Q^{n+1}_{1,2,\ldots,i,\ldots,k,\ldots,n+1}u$ and $Q^{n+1}_{1,2,\ldots,k,\ldots,i,\ldots,n+1}u$, together with q. c.'s of lower orders, i. e. we get the law desired.

Case II.— When $i = 1$, $k \neq n+1$.

Put $Q^{n-k+1}_{k+1,\ldots,n+1}u = v$, then $Q^{n+1}_{i,2,\ldots,k,\ldots,n+1}u = Q^{k}_{i,2,\ldots,k}v$. The relation between $Q^{k}_{i,2,\ldots,k}v$ and $Q^{k}_{k,2,\ldots,i}v$ is known, k being at most equal to n, hence is also the relation between $Q^{n+1}_{i,2,\ldots,k,\ldots,n+1}u$ and $Q^{n+1}_{k,2,\ldots,i,\ldots,n+1}u$.

Case III.— When $i = 1$, $k = n+1$.

Consider any third variable as x_j. We first establish the law for the interchange of x_j and x_k by case I. This gives $Q^{n+1}_{i,2,\ldots,j,\ldots,k}u$ in terms of $Q^{n+1}_{i,2,\ldots,k,\ldots,j}u$. Then we express $Q^{n+1}_{i,2,\ldots\ ,k,\ldots,j}u$ in terms of $Q^{n+1}_{k,2,\ldots,i,\ldots,j}u$ by case II. Applying now case I again we express $Q^{n+1}_{k,2,\ldots,i,\ldots,j}u$ in terms of $Q^{n+1}_{k,2,\ldots,j,\ldots,i}u$. Between the equations expressing these relations we eliminate $Q^{n+1}_{i,2,\ldots,k,\ \ ,j}u$ and $Q^{n+1}_{k,2,\ldots,i,\ldots,j}u$ and obtain an equation between $Q^{n+1}_{i,2,\ldots j,\ldots,k}u$, $Q^{n+1}_{k,2,\ldots j,\ldots,i}u$ and q. c's of a lower order than $n+1$. Moreover this equation is rational, and linear in the q. c's of the $(n+1)^{\text{th}}$ order.

We conclude then, that knowing the law for the interchange of variables in a q. c. of one order we can deduce the law for the interchange of variables in

the q. c's of the next higher, and hence of any higher order, and having already established these laws for the q. c's of the second and third orders, the problem is theoretically solved. The general formula can not be compactly exhibited, but can be readily worked out for any special case. Applied to the possible cases of the various quotiential coefficients of the fourth order, we have

$$Q_{1234}u + Q_{134}u = Q_{1324}u + Q_{124}u,$$
$$Q_{1234}u \cdot Q_{234}u = Q_{1234}u \cdot Q_{134}u,$$
$$[Q_{1234}u + Q_{134}u]\, Q_{324}u = [Q_{3214}u + Q_{314}u]\, Q_{124}u,$$
$$Q_{1234}u \cdot Q_{234}u + Q_{124}u \cdot Q_{24}u = Q_{1243}u \cdot Q_{243}u + Q_{123}u \cdot Q_{23}u,$$
$$[Q_{1234}u + Q_{143}u]\, Q_{234}u + [Q_{124}u + Q_{143}u]\, Q_{24}u = [Q_{1432}u + Q_{123}u]\, Q_{432}u + [Q_{142}u + Q_{123}u]\, Q_{42}u,$$
$$\{Q_{1234}u \cdot Q_{234}u + Q_{124}u \cdot Q_{24}u + [Q_{13}u + Q_{143}u]\, Q_{243}u\}\, Q_{423}u \cdot Q_{213}u = \{Q_{4231}u \cdot Q_{231}u + Q_{421}u \cdot Q_{21}u + [Q_{43}u + Q_{413}u]\, Q_{213}u\}\, Q_{123}u \cdot Q_{243}u.$$

It will be observed that *the right member of each of the above equations can be written down from the corresponding left members, by simply interchanging the suffixes i and k of the variables x_i and x_k, and it can be shown that this is true generally.*

9.—SUCCESSIVE QUOTIENTIATION.

We now return to functions of a single variable x. As a rule, the n^{th} q. c. of $y = f(x)$ does not admit of a simple algebraic expression, but there is at least one exception. For $y = \log x$, $\frac{qy}{qx} = \frac{1}{\log x}$, $\frac{q^2y}{qx^2} = \frac{-1}{\log x}$, and after that, every successive q. c. is equal to $\frac{-1}{\log x}$, so that

$$\frac{q^n(\log x)}{qx^n} = \pm \frac{1}{\log x},$$

according as $n = 1$ or $n \neq 1$.

The second q. c. can in most cases be readily expressed. If $y = \log^n x$, we have

$$\frac{qy}{qx} = \frac{1}{\log^n x} \cdot \frac{1}{\log^{n-1} x} \cdots \frac{1}{\log x},$$

and

$$\frac{q^2y}{qx^2} = -\left(\frac{q \log^n x}{qx} + \frac{q \log^{n-1} x}{qx} + \cdots + \frac{q \log x}{qx}\right).$$

If $y = u.v.w \ldots$, $u, v, w, \ldots$ being functions of x,

$$\frac{qy}{qx} = \sum \frac{qu}{qx} \quad \text{and} \quad \frac{q^2y}{qx^2} = \sum \frac{qu}{qx} \frac{q^2u}{qx^2} \Big/ \sum \frac{qu}{qx}.$$

If $y = u + v + w + \ldots$,

$$\frac{qy}{qx} = \frac{\sum u \frac{qu}{qx}}{\sum u}, \quad \frac{q^2y}{qx^2} = \frac{\sum u \frac{q^2u}{qx^2} + \sum u \frac{qu}{qx} \frac{q^2u}{qx^2}}{\sum u \frac{qu}{qx}} - \frac{\sum u \frac{qu}{qx}}{\sum u}$$

Using the notation introduced in the preceding paragraph, these formulæ may be written:

For $y = u.v.w \ldots$

$$Qy = \Sigma Qu, \quad Qy \,.\, Q^2y = \Sigma Qu\, Q^2u;$$

For $y = u + v + w + \ldots$

$$yQy = \Sigma uQu, \quad yQyQ^2y + y\overline{Qy}^2 = \Sigma uQuQ^2u + \Sigma u\overline{Qu}^2.$$

Let us find an equation involving the third q. c. of $y = u + v + w + \ldots$. To do this we quotientiate the expression

$$yQyQ^2y + y\overline{Qy}^2 = \Sigma uQuQ^2u + \Sigma u\overline{Qu}^2,$$

and obtain

$$\frac{yQyQ^2y\,(Qy + Q^2y + Q^3y) + y\overline{Qy}^2(Qy + 2\,Q^2y)}{yQyQ^2y + y\overline{Qy}^2} = \frac{\Sigma uQuQ^2u\,(Qu + Q^2u + Q^3u) + \Sigma u\overline{Qu}^2(Qu + 2\,Q^2u)}{\Sigma uQuQ^2u + \Sigma u\overline{Qu}^2},$$

and cancelling the equal denominators and collecting like terms,

$$y\overline{Qy}^3 + 3y\overline{Qy}^2Q^2y + yQy\overline{Q^2y}^2 + yQyQ^2yQ^3y = \sum u\overline{Qu}^3 + 3\sum u\overline{Qu}^2Q^2u + \sum uQu\overline{Q^2u}^2 + \sum uQuQ^2uQ^3u.$$

An examination of this expression involving Q^3y, and a comparison of it with the similar expressions involving Q^2y and Qy respectively, suggests the following law of formation:

(a). *The left member of the equation is homogeneous of the fourth degree in* y, Qy, Q^2y, Q^3y.

(b). *Each term contains the factor y and a cofactor of degree three, which is a multiple of Qy, and consecutive ones of the quantities, Qy, Q^2y, Q^3y.*

(c). *All the terms, 2^2 in number, that can thus be formed, occur.*

(d). *The sum of all the coefficients on the left is* 3 !

(e). *The right side of the equations can be formed from the left side by replacing every term on the left by a sum of terms formed by substituting in that term for y successively the quantities u, v, w, etc. and adding the results.*

We now show that these statements can be generalized for an expression involving Q^ny and quotiential coefficients of lower orders. To do this we assume the existence of an expression formed according to the above law involving $Q^{n-1}y$ and quotiential coefficients of lower orders. Quotientiating this gives an expression, involving Q^ny and lower quotientials, which obeys the same law. But we have shown the existence of such an expression when $n-1=3$, hence the law holds in general.

(a). *The terms on the left are homogeneous of the $n+1^{\text{th}}$ degree in y, Qy, Q^2y, $\ldots$, Q^ny, where n is the order of the highest quotiential which is involved.*

Notice, first, that since the q. c. of a sum of terms is equal to a sum of products formed by multiplying each term by its q. c., adding the results and dividing the sum of the partial products by the original sum, the divisors on the two sides of the equation, after quotientiation, will cancel, being equal. The quotientiation of a sum of terms equal to another sum reduces then to a consideration of products of the individual terms by their quotientials.

Let

$$Ay\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\ldots\overline{Q^sy}^{\sigma},\quad \alpha+\beta+\ldots+\sigma=n-1,\quad s\leqq n-1,$$

be a term of the assumed equation. Multiplying this by its q. c., we obtain

$$Ay\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\ldots\overline{Q^sy}^{\sigma}(Qy+\alpha Q^2y+\beta Q^3y+\ldots+\sigma Q^{s+1}y).$$

This shows that the degree of every term is raised by unity, and hence, if the assumed expression is homogeneous of degree n, the expression obtained by quotientiation must be homogeneous and of degree $n+1$. A new factor Q^ny is introduced in the case, $s=n-1$, so that the new expression involves y, Qy, Q^2y, $\ldots$, Q^ny.

(b). *Each term contains the factor y, and a cofactor of degree n, which is some multiple of Qy and consecutive ones of the quantities Qy, Q^2y, $\ldots$, Q^ny.*

That each term contains the factor y is obvious when we remember that each term of the result of the quotientiation contains some term of the original expression as a factor. Again, every term except the last arising from the quotientiation of a single term contains all the constituent factors of that term, while the last term contains, besides the original constituents Qy, Q^2y, ..., Q^sy, the new constituent $Q^{s+1}y$.

(c). *All the terms of degree* $n+1$, 2^{n-1} *in number, that can be formed to satisfy* (*b*), *occur.*

Every term of the $(n+1)^{\text{th}}$ degree can be formed from some term of the n^{th} degree, by either increasing the exponent of some Q by unity, or by adding the factor Q^ny. Both possibilities are exhausted during the process of quotientiation. Hence, all possible terms occur in the expression of the $(n+1)^{\text{th}}$ degree provided all possible terms exist in the assumed expression.

The total number of these terms is 2^{n-1}, for, to every term in the assumed expression as

$$y\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\ldots.\ \overline{Q^sy}^{\sigma},\quad 1+\alpha+\beta+\ldots.+\sigma=n,\ s\leqq n-1,$$

correspond two terms in the final expression, viz.

$$yQy\overline{Q^2y}^{\alpha}\overline{Q^3y}^{\beta}\ldots.\ \overline{Q^{s+1}y}^{\sigma},\quad 1+1+\alpha+\beta+\ldots.+\sigma=n+1,\ s+1\leqq n,\ \text{[A]}$$

and

$$y\overline{Qy}^{\alpha+1}\overline{Q^2y}^{\beta}\ldots.\ \overline{Q^sy}^{\sigma},\quad 1+(\alpha+1)+\beta+\ldots.+\sigma=n+1,\ s<n,\qquad \text{[B]}$$

It is clear that the terms thus derived are all different. That the totality of these pairs of terms exhausts all possible terms in the final expression is evident, since any term in the final expression as

$$y\overline{Qy}^{\alpha'}\overline{Q^2y}^{\beta'}\ldots.\ \overline{Q^ry}^{\sigma'},\quad 1+\alpha'+\beta'+\ldots.+\sigma'=n+1,\quad r\leqq n,$$

is of the form [A] or [B] according as $\alpha=1$ or $\alpha\neq 1$. Finally, because of this one-to-two correspondence between the 2^{n-2} terms of the assumed expression, and those of the result of quotientiation, the number of terms in the latter is $2^{n-2}.2$ or 2^{n-1}.

(d). *The sum of all the coefficients on the left is* $n!$

Let one term of the original expression be $Ay\,\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\ldots.\ \overline{Q^sy}^{\sigma}$. Quotientiating this, we get, for corresponding term in the numerator,

$$Ay\,\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\ldots.\ \overline{Q^sy}^{\sigma}(Qy+\alpha Q^2y+\beta Q^3y+\ldots.+\sigma Q^{s+1}y),$$

the sum of the coefficients of which is $A(1+\alpha+\beta+\ldots+\sigma)=An$. Similarly for every other term. The original coefficient sum $n-1$! is hence multiplied by n, giving $n!$ for the final coefficient sum.

(e). *The right side of the equation can be formed from the left side by replacing every term on the left by a sum of terms, formed by substituting in that term for y successively the quantities u, v, w, etc., and adding the results.*

To any one term in the assumed expression, as $Ay\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\ldots\,\overline{Q^sy}^{\sigma}$, there corresponds on the right-hand side a sum of terms $A\Sigma u\,\overline{Qu}^{\alpha}\,\overline{Q^2u}^{\beta}\ldots\,\overline{Q^su}^{\sigma}$. Consider the terms that are yielded by these corresponding terms on quotientiation. The left term yields

$$Ay\,\overline{Qy}^{\alpha+1}\,\overline{Q^2y}^{\beta}\ldots\,\overline{Q^sy}^{\sigma}+\alpha Ay\,\overline{Qy}^{\alpha}\,\overline{Q^ry}^{\beta+1}\ldots\,\overline{Q^sy}^{\sigma}+\ldots\ldots+\sigma Ay\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\ldots\,Q^{s+1}y,$$

while the corresponding right side sum yields

$$\begin{aligned}&Au\,\overline{Qu}^{\alpha}\,\overline{Q^2u}^{\beta}\ldots\,\overline{Q^su}^{\sigma}(Qu+\alpha Q^2u+\beta Q^3u+\ldots+\sigma Q^{s+1}u)\\+\;&Av\,\overline{Qv}^{\alpha}\,\overline{Q^rv}^{\beta}\ldots\,\overline{Q^sv}^{\sigma}(Qv+\alpha Q^2v+\beta Q^3v+\ldots+\sigma Q^{s+1}v)\\+\;&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\+\;&\text{etc.}\end{aligned}$$

This, by summing terms which stand under each other, may be written

$$A\Sigma u\,\overline{Qu}^{\alpha+1}\,\overline{Q^2u}^{\beta}\ldots\,\overline{Q^su}^{\sigma}+\alpha A\Sigma u\,\overline{Qu}^{\alpha}\,\overline{Q^2u}^{\beta+1}\ldots\,\overline{Q^su}^{\sigma}+\ldots\ldots+\sigma A\Sigma u\,\overline{Qu}^{\alpha}\,\overline{Q^2u}^{\beta}\ldots\,Q^{s+1}u.$$

We see then that the law expressed in (e) holds for the results of corresponding terms, hence also for the collected results.

We have now determined everything except the specific coefficients of the individual terms. To find these, let us consider the general term of the expression giving Q^ny, viz.

$$A_n y\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\,\overline{Q^3y}^{\gamma}\ldots\,\overline{Q^ry}^{\rho}\,\overline{Q^sy}^{\sigma},\quad \alpha+\beta+\gamma+\ldots+\rho+\sigma=n.$$

This term is obtained by multiplying each of the following terms belonging to

the expression for $Q^{n-1}y$ by its respective q. c.,

$$\begin{aligned}
&A_{n-1}y\overline{Qy}^{\alpha-1}\overline{Q^2y}^{\beta}\overline{Q^3y}^{\gamma}\ \ldots\ \overline{Q^ry}^{\rho}\overline{Q^sy}^{\sigma} \text{ by } Qy,\\
&B_{n-1}y\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta-1}\overline{Q^3y}^{\gamma}\ \ldots\ \overline{Q^ry}^{\rho}\overline{Q^sy}^{\sigma} \text{ " } \alpha Q^2y,\\
&C_{n-1}y\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\overline{Q^3y}^{\gamma-1}\ \ldots\ \overline{Q^ry}^{\rho}\overline{Q^sy}^{\sigma} \text{ " } \beta Q^3y,\\
&\vdots\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots\\
&S_{n-1}y\overline{Qy}^{\alpha}\overline{Q^2y}^{\beta}\overline{Q^3y}^{\gamma}\ \ldots\ \overline{Q^ry}^{\rho}\overline{Q^sy}^{\sigma-1} \text{ " } \rho Q^sy.
\end{aligned}$$

Adding these products and collecting the coefficients, we see that

$$A_n = A_{n-1} + \alpha B_{n-1} + \beta C_{n-1} + \ldots + \rho S_{n-1},$$

a formula which enables us to compute the coefficients of the expression for Q^ny from those of the expression for $Q^{n-1}y$. The computation of the complete expression for Q^ny becomes laborious when n is large. The properties set forth under a), b), c), d) and e) furnish efficient checks for the computation. Below we give the expressions for $n = 4$, 5 and 6.

$n = 4$.

$$\begin{aligned}
&y\overline{Qy}^{4} + 6y\overline{Qy}^{3}Q^2y + 7y\overline{Qy}^{2}\overline{Q^2y}^{2} + 4y\overline{Qy}^{2}Q^2yQ^3y + yQy\overline{Q^2y}^{3} + 3yQy\overline{Q^2y}^{2}Q^3y\\
&\qquad + yQyQy^2\overline{Q^3y}^{2} + yQyQ^2yQ^3yQ^4y = \Sigma u\overline{Qu}^{4} + \textit{etc.}
\end{aligned}$$

$n = 5$.

$$\begin{aligned}
&y\overline{Qy}^{5} + 10y\overline{Qy}^{4}Q^2y + 25y\overline{Qy}^{3}\overline{Q^2y}^{2} + 10y\overline{Qy}^{3}Q^2yQ^3y + 15yQy\overline{Q^2y}^{3} + 25y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2}\\
&\quad + 5y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2} + 5y\overline{Qy}^{2}Q^2yQ^3yQ^4y + yQy\overline{Q^2y}^{4} + 6yQy\overline{Q^2y}^{3}Q^3y\\
&\quad + 7yQy\overline{Q^2y}^{2}\overline{Q^3y}^{2} + 4yQy\overline{Q^2y}^{2}Q^3yQ^4y + yQyQ^2y\overline{Q^3y}^{3}\\
&\quad + 3yQyQ^2y\overline{Q^3y}^{2}Q^4y + yQyQ^2yQ^3y\overline{Q^4y}^{2} + yQyQ^2yQ^3yQ^4yQ^5y\\
&\quad = \Sigma u\overline{Qu}^{5} + \textit{etc.}
\end{aligned}$$

$n = 6$.

$$\begin{aligned}
&y\overline{Qy}^{6} + 15y\overline{Qy}^{5}Q^2y + 65y\overline{Qy}^{4}\overline{Q^2y}^{2} + 20y\overline{Qy}^{4}Q^2y\,Q^3y + 90\,y\overline{Qy}^{3}\overline{Q^2y}^{3}\\
&\quad + 105y\overline{Qy}^{3}Q^2y\overline{Q^3y}^{2} + 15y\overline{Qy}^{3}\overline{Q^2y}^{2}Q^3y + 15y\overline{Qy}^{3}Q^2y\,Q^3y\,Q^4y + 31y\overline{Qy}^{2}\overline{Q^2y}^{4}\\
&\quad + 101y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{3} + 67y\overline{Qy}^{2}\overline{Q^2y}^{2}\overline{Q^3y}^{2} + 39y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2}Q^4y\\
&\quad + 6y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{3} + 18y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2}Q^4y + 6y\overline{Qy}^{2}Q^2y\,Q^3y\overline{Q^4y}^{2}\\
&\quad + 6y\overline{Qy}^{2}Q^2y\,Q^3y\,Q^4y\,Q^5y + y\,Qy\overline{Q^2y}^{5} + 10y\,Qy\overline{Q^2y}^{4}Q^3y + 25y\,Qy\overline{Q^2y}^{3}\overline{Q^3y}^{2}\\
&\quad + 10y\,Qy\overline{Q^2y}^{3}Q^3y\,Q^4y + 15y\,Qy\overline{Q^2y}^{2}\overline{Q^3y}^{3} + 25y\,Qy\overline{Q^2y}^{2}\overline{Q^3y}^{2}Q^4y\\
&\quad + 5y\,Qy\overline{Q^2y}^{2}Q^3y\overline{Q^4y}^{2} + 5y\,Qy\overline{Q^2y}^{2}Q^3y\,Q^4y\,Q^5y + y\,Qy\,Q^2y\overline{Q^3y}^{4}\\
&\quad + 6y\,Qy\,Q^2y\overline{Q^3y}^{3}Q^4y + 7y\,Qy\,Q^2y\overline{Q^3y}^{2}\overline{Q^4y}^{2} + 4y\,Qy\,Q^2y\,Q^3y\overline{Q^4y}^{2}Q^5y\\
&\quad + y\,Qy\,Q^2y\,Q^3y\overline{Q^4y}^{3} + 3y\,Qy\,Q^2y\,Q^3y\overline{Q^4y}^{2}Q^5y + y\,Qy\,Q^2y\,Q^3y\,Q^4y\overline{Q^5y}^{2}\\
&\quad + y\,Qy\,Q^2y\,Q^3y\,Q^4y\,Q^5y\,Q^6y = \Sigma u\overline{Qu}^{6} + etc.
\end{aligned}$$

It is now easy to find the successive q. c's of a product. For if

$$y = u.v.w\ldots.,$$

then

$$Qy = Qu + Qv + Qw + \ldots.,$$

so that, in order to form the expression for the n^{th} q. c. of a product, we need only to substitute in the expression for the $n-1^{\text{th}}$ q. c. of a sum

$$Qy,\ Qu,\ Qv, \text{ etc.}, \quad \text{for} \quad y,\ u,\ v, \text{ etc.},$$

and generally

$$Q^{r+1}y, Q^{r+1}u,\ Q^{r+1}v, \text{ etc.}, \quad \text{for} \quad Q^ry,\ Q^ru,\ Q^rv, \text{ etc.}$$

Thus, if $y = u.v.w\ldots.$, we have for $n = 3$ and $n = 4$ respectively

$$Qy\,\overline{Q^2y}^{3} + 3\,Qy\,\overline{Q^2y}^{2}\,Q^3y + Qy\,Q^2y\,\overline{Q^3y}^{2} + Qy\,Q^2y\,Q^3y\,Q^4y = \Sigma\,Qu\,\overline{Q^2u}^{3} + \text{etc.},$$

$$\begin{aligned}
&Qy\,\overline{Q^2y}^{4} + 6\,Qy\,\overline{Q^2y}^{3}\,Q^3y + 7\,Qy\,\overline{Q^2y}^{2}\,\overline{Q^3y}^{2} + 4\,Qy\,\overline{Q^2y}^{2}\,Q^3y\,Q^4y + Qy\,Q^2y\,\overline{Q^3y}^{3}\\
&\quad + 3\,Qy\,Q^2y\,\overline{Q^3y}^{2}\,Q^4y + Qy\,Q^2y\,Q^3y\,\overline{Q^4y}^{2} + Qy\,Q^2y\,Q^3y\,Q^4y\,Q^5y = \Sigma\,Qu\,\overline{Q^2u}^{4} + \text{etc.},
\end{aligned}$$

10.—SYZYGIES CONNECTING DIFFERENTIAL AND QUOTIENTIAL PROCESSES.

It remains to consider operations into which differential and quotiential processes enter conjointly. In discussing such, it is convenient to use the follow-

ing notation:

$$Dy = \frac{dy}{dx}, \quad QDy = \frac{q(Dy)}{qx}, \quad DQy = \frac{d(Qy)}{dx},$$

similarly,

$$DQDy = \frac{d(QDy)}{dx}, \quad D^2Qy = \frac{d(DQy)}{dx}, \quad QDQy = \frac{q(DQy)}{qx}, \text{ etc.}$$

It is now easy to express these various mixed operations in terms of successive q. c's alone. We have

I. 1. $Dy = \frac{y}{x}\, Qy,$

II. 2. $QDy = Q\left(\frac{y}{x}\, Qy\right) = Q^2y + Qy - 1,$

similarly,

3. $DQy = \frac{1}{x}\, Qy\, Q^2y,$

4. $D^2y = \frac{y}{x^2}\, Qy\,(Q^2y + Qy - 1),$

III. 5. $Q^2Dy = \frac{Q^2y\,(Q^3y + Qy)}{Q^2y + Qy - 1},$

6. $QDQy = Q^3y + Q^2y - 1,$

7. $QD^2y = \frac{Q^2y\,(Q^3y + Qy) - 1}{Q^2y + Qy - 1} + 1,$

8. $DQ^2y = \frac{Q^2y\,Q^3y}{x},$

9. $DQDy = \frac{Q^2y}{x}\,(Q^3y + Qy),$

10. $D^2Qy = \frac{Qy\,Q^2y}{x^2}\,(Q^3y + Q^2y - 1),$

11. $D^3y = \frac{y}{x^3}\, Qy\,(\overline{QD}^2y - QDy + xDQDy)$

$$= \frac{y}{x^3}\, Qy\,[(Q^2y + Qy - 1)^2 - (Q^2y + Qy - 1) + (Q^3y\,Q^2y + Q^2y\,Qy)].$$

These mixed operations may also be expressed in terms of differential processes alone, but in most cases the resulting formulæ are more complex than

those given above. For instance, 6) and 8) become respectively

$$QDQy = \frac{x}{y} \cdot \frac{2x\overline{Dy}^{3} - 2y\overline{Dy}^{2} - 3xyDyD^2y + 2y^2D^2y + xy^2D^3y}{yDy + xyD^2y - x\overline{Dy}^{2}},$$

$$DQ^2y = \frac{x\overline{Dy}^{4} - y\overline{Dy}^{3} - xy\overline{DyD^2y} + y^2DyD^2y + xy^2DyD^3y - xy^2\overline{D^2y}^{2}}{y^2\overline{Dy}^{2}}.$$

Each of the groups of operations, I, II, III, is connected by a remarkable syzygy, which affords a convenient check in the computation of successive groups.

I. We may write

$$\frac{Dy}{Qy} \cdot \frac{x}{y} = 1.$$

If we eliminate Qy and $\overline{Qy}^{2}$ from the equations under II, we have

$$\frac{D^2y}{QDy} \cdot \frac{Q^2y}{DQy} \cdot \frac{x}{y} = 1,$$

Eliminating Qy and Q^2y from the equations under III, we find

$$\frac{D^3y}{Q^3y} \cdot \frac{Q^2Dy}{D^2Qy} \cdot \frac{DQ^2y}{QD^2y} \cdot \frac{QDQy}{DQDy} \cdot \frac{x}{y} = 1,$$

and similarly, if we should use the groups of equations into which respectively four and five successive operations enter, we would find the syzygies,

$$\frac{D^4y}{QD^3y} \cdot \frac{Q^4y}{DQ^3y} \cdot \frac{D^2Q^2y}{QDQ^2y} \cdot \frac{Q^2D^2y}{DQD^2y} \cdot \frac{DQ^2Dy}{Q^3Dy} \cdot \frac{QD^2Qy}{D^3Qy} \cdot \frac{DQDQy}{Q^2DQy} \cdot \frac{QDQDy}{D^2QDy} \cdot \frac{x}{y} = 1,$$

and

$$\frac{D^5y}{Q^5y} \cdot \frac{D^3Q^2y}{Q^3D^2y} \cdot \frac{D^2Q^2Dy}{Q^2D^2Qy} \cdot \frac{D^2QDQy}{Q^2DQDy} \cdot \frac{DQ^4y}{QD^4y} \cdot \frac{DQ^2D^2y}{QD^2Q^2y} \cdot \frac{DQD^2Qy}{QDQ^2Dy} \cdot \frac{DQDQDy}{QDQDQy} \cdot \frac{Q^2D^3y}{D^2Q^3y}$$
$$\cdot \frac{QDQ^3y}{DQD^3y} \cdot \frac{Q^2DQ^2y}{D^2QD^2y} \cdot \frac{QDQD^2y}{DQDQ^2y} \cdot \frac{Q^4Dy}{D^4Qy} \cdot \frac{QD^3Qy}{DQ^3Dy} \cdot \frac{Q^3DQy}{D^3QDy} \cdot \frac{QD^2QDy}{DQ^2DQy} \cdot \frac{x}{y} = 1,$$

From these special cases a general law can readily be deduced. In fact if $F_1, F_2, \ldots, F_n$, represent the factors in the numerator, and $f_1, f_2, \ldots, f_n$, the factors of the denominator of the syzygy connecting operations of any order, then

$$\frac{\prod_{i=1}^{n}(DF_i \,.\, Qf_i)}{\prod_{j=1}^{n}(QF_j \,.\, Df_j)} = 1$$

represents the syzygy connecting the operations of the next higher order.

11.—THE PROCESS $)_3$ IN THE ORDINARY ALGEBRA.

We have shown how a consistent calculus can be built up in which the process of quotientiation occupies the same place that the process of differentiation occupies in the ordinary calculus. This calculus is not only consistent but could be used, though with less ease than the differential calculus, to represent many facts of the universe. For just as it is arbitrary whether we represent aggregates by sums, or products, or powers of certain of their constituents, so likewise is it arbitrary whether we consider growth as accretion or expansion, and change of any kind as caused by additive or multiplicative processes. In the one case, we give to the element which is considered independent in its change, an additive increment dx, in the other case we give it a multiplicative expansion qx. While it may be convenient in practice to measure absolute change by subtracting the initial state from the final, and relative change by division, it is not necessary to do this. We could measure all primary change by dividing the final state by the initial, and relative change by computing the logarithm of the final state with reference to the initial as base. If logarithms not ratios, powers not multiples, were the functions sought in practice, quotientiation would logically take the place of differentiation, and would probably have preceded it in the evolution of mathematical knowledge.

Let us next consider whether a consistent and possible calculus could be built up involving the process $\frac{d_2y}{d_2x} \cdot \frac{d_2y}{d_2x}$ was defined as $\lim_{h=M_2} \{F(x)_2 h) \frown_2 F(x) \frown_3 h\}$, an expression which involves the symbol $\frown_3$. The ordinary algebra is not usually extended beyond the process $)_2$ and its inverses, that is, beyond involution and its inverses, evolution and the finding of logarithms, so that the consideration of $\frac{d_2y}{d_2x}$ necessitates a short preliminary discussion of the process $)_3$ and its inverses.

The equation $\quad a)_2 a)_2 \ldots$. to b terms $= a)_3 b$,

when applied to the ordinary algebra, and written at length, becomes $\left(\left(\ldots\left((a)^a\right)^a\ldots\right)^a\right)^a$ and we shall denote this by $a^{\underline{|b}}$, where b is the number of a's which are involved. If we interpret generally $a^{\underline{|b-1}}$ to be that function

which, when raised to the a^{th} power, gives $a\underline{\lfloor b}$, we have immediately $a\underline{\lfloor 1} = a$, $a\underline{\lfloor 0} = \sqrt[a]{a}$, $a\underline{\lfloor -1} = \sqrt[a]{\sqrt[a]{a}}$, and generally,

$$a\underline{\lfloor -b} = \sqrt[a]{\sqrt[a]{\ldots\sqrt[a]{a}}} = \sqrt[a^{b-1}]{a},$$

where the expression in the middle contains $(b+1)$ radicals, that is, $(b+2)$ letters a.

The first inverse $x = a \frown_3 b$ is defined by $x\underline{\lfloor b} = a$, and we shall denote the x thus defined by $x = \underline{b\rfloor} a$. The second inverse $x = a \frown_3 b$ is defined by $b\underline{\lfloor x} = a$, and the x satisfying this equation we denote by $x = rg_b a$, which we read x equals range of a to the scale b. We have then for the complete definition equations of the first and second inverses of $a\underline{\lfloor b}$ respectively

$$(\underline{b\rfloor} a)\underline{\lfloor b} = a \quad \text{and} \quad b\underline{\lfloor rg_b a} = a.$$

$a\underline{\lfloor b}$ may also be written $a^{a^{b-1}}$, and using this relation, we may easily establish the following rules of operation:

(a). $(x.y)\underline{\lfloor n} = (x\underline{\lfloor n})^{y^{n-1}}.(y\underline{\lfloor n})^{x^{n-1}}$,

(b). $(x^y)\underline{\lfloor n} = (x\underline{\lfloor (n-1)y+1})^y$,

(c). $(x\underline{\lfloor y})\underline{\lfloor n} = x\underline{\lfloor (n-1)x^{y-1}+y}$,

(d). $(a\underline{\lfloor x+y} = (a\underline{\lfloor x})^{a^y} = (a\underline{\lfloor y})^{a^x}$,

(e). $\log_a a\underline{\lfloor xy} = a^{xy-1}$,

(f). $\log_a (a\underline{\lfloor x+y}) = a \log_a (a\underline{\lfloor x}) . \log_a (a\underline{\lfloor y}) = \dfrac{a^x . a^y}{a}$,

(g). $\log_a (a\underline{\lfloor x} . a\underline{\lfloor y})^a = a^x + a^y$,

(h). $\log_{a\underline{\lfloor n}} (a\underline{\lfloor m}) = a^{m-n}$.

The differential and quotiential coefficients may now be readily computed The work is so simple that we shall give only results.

(i) $y = x\underline{\lfloor a}$, $\dfrac{dy}{dx} = yx^{a-2}(1 + \overline{a-1}\log x)$, $\dfrac{qy}{qx} = x^{a-1}(1 + \overline{a-1}\log x)$,

(j) $y = a\underline{\lfloor x}$, $\dfrac{dy}{dx} = y \log a \log y$, $\dfrac{qy}{qx} = x \log a \log y$.

(k) $y = \underline{a\rfloor} x$, $\dfrac{dy}{dx} = \dfrac{y \log y}{x \log x} \cdot \dfrac{1}{1+(a-1)\log y}$, $\dfrac{qy}{qx} = \dfrac{\log y}{\log x} \cdot \dfrac{1}{1+(a-1)\log y}$,

(l) $y = \underline{x\rfloor} a$, $\dfrac{dy}{dx} = \dfrac{y\overline{\log y}^2}{\log\log y - rg_e a}$, $\dfrac{qy}{qx} = \dfrac{x\overline{\log y}^2}{\log\log y - rg_e a}$,

(m) $y = rg_a x$, $\dfrac{dy}{dx} = \dfrac{1}{x \log x \log a}$, $\qquad \dfrac{qy}{qx} = \dfrac{1}{y \log x \log a}$,

(n) $y = rg_x a$, $\dfrac{dy}{dx} = -\dfrac{1 + (y-1)\log x}{x \overline{\log x}^2}$, $\qquad \dfrac{qy}{qx} = -\dfrac{1 + (y-1)\log x}{y \overline{\log x}^2}$,

12.—THE LIMITING PROCESS DENOTED BY $\frac{ry}{rx}$.

$a \underline{\lfloor 1} = a$, hence the equation $a \,)_2 M_2 = a$ is satisfied and by theorem I, Art. 2,

$$\frac{d_2 y}{d_2 x} = \lim_{h = M_2} \{F(x \,)_2 h) \frown_2 F(x) \frown_3 h\} = M_2 \frown_3 M_2$$

is indeterminate. When $)_2$, $\frown_2$, $\frown_3$ and M_2 are replaced by their algebraic equivalents, we denote this limit by $\dfrac{ry}{rx}$, so that

$$\frac{ry}{rx} = \lim_{h=1} \{rg_h \log_{F(x)} F(x^h)\},$$

$$= 1 + \lim_{h=1} \{\log_h \log_h \log_{F(x)} F(x^h)\}. \qquad [A]$$

$$= 1 + \lim_{h=1} \left\{ \frac{rg_e \log_{F(x)} F(x^h) - rg_e h}{\log h} \right\}. \qquad [B]$$

Either of the last two forms is convenient for the evaluation of the indeterminate. We consider some special cases first.

(a). $y = x^n$. By formula [B],

$$\frac{ry}{rx} = 1 + \lim_{h=1} \left\{ \frac{rg_e \log_{x^n} (x^h)^n - rg_e h}{\log h} \right\} = 1 + \lim_{h=1} \left\{ \frac{rg_e h - rg_e h}{\log h} \right\} = 1.$$

(b). $y = x \underline{\lfloor n}$. By formula [A],

$$\frac{ry}{rx} = 1 + \lim_{h=1} \{\log_h \log_h \log_{x \underline{\lfloor n}} (x^h) \underline{\lfloor n}\}.$$

Put $x = e^z$, then $\log_{x \underline{\lfloor n}} (x^h) \underline{\lfloor n} = \log (e^{hz}) \underline{\lfloor n} / \log (e^z) \underline{\lfloor n}$.

Now, by formula (b), sect. 11,

$$(e^{hz}) \underline{\lfloor n} = (e^{\underline{\lfloor (n-1) hz + 1}})^{hz} \quad \text{and} \quad (e^z) \underline{\lfloor n} = (e^{\underline{\lfloor (n-1) z + 1}})^z,$$

so that $\log (e^{hz}) \underline{\lfloor n} = hze^{(n-1) hz}$ and $\log (e^z) \underline{\lfloor n} = ze^{(n-1) z}$.

Hence,

$$\frac{ry}{rx} = 1 + \lim_{h=1} \left\{ \log_h \log_h \left(\frac{hze^{(n-1) hz}}{ze^{(n-1) z}} \right) \right\},$$

but
$$\lim_{h=1}\left\{\log_h \frac{hze^{(n-1)hz}}{ze^{(n-1)z}}\right\} = \frac{q\left(ze^{(n-1)z}\right)}{qx} = \frac{qz}{qz} + \frac{q\left(e^{(n-1)z}\right)}{qz}$$
$$= 1+(n-1)z = 1+(n-1)\log x,$$

so that finally
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h\left(1 + \overline{n-1}\log x\right)\right\} = 1+\infty = \infty,$$

unless $n = 1$, the case which has already been considered.

(c). $y = e^x$. By [A]
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log_h \log_{e^x}\left(e^{x^h}\right)\right\}$$
$$\log_{e^x} e^{x^h} = \log_{e^x}\left(e^x\right)^{x^{h-1}} = x^{h-1},$$

hence
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log_h\left(x^{h-1}\right)\right\}.$$

Put $x = e^z$, then $x^{h-1} = e^{zh}/e^z$,

and
$$\lim_{h=1}\left\{\log_h x^{h-1}\right\} = \lim_{h=1}\left\{\log_h \frac{e^{zh}}{e^z}\right\} = \frac{q(e^z)}{qx} = z = \log x,$$

so that finally
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log x\right\} = 1+\infty = \infty.$$

In each of the foregoing examples except the first, we found $\frac{ry}{rx} = \infty$. We shall now prove that $y = x^n$ is the only possible function of x for which $\frac{ry}{rx}$ is finite.

Let $y = f(x)$. We have
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log_h \log_{f(x)} f(x^h)\right\},$$

which, on putting $x = e^z$, becomes
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log_h \frac{\log f(e^{zh})}{\log f(e^z)}\right\}.$$

now
$$\lim_{h=1}\left\{\log_h \frac{\log f(e^{zh})}{\log f(e^z)}\right\} = \frac{q\left(\log f(e^z)\right)}{qz},$$

so that
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h\left[\frac{q\left(\log f(e^z)\right)}{qz}\right]\right\},$$

and since $\log_1 a = \infty$, $a \neq 1$, this expression $= \infty$ unless $\frac{q \log f(e^z)}{qz} = 1$. But

$$\frac{q \log f(e^z)}{qz} = \frac{q \log f(e^z)}{qf(e^z)} \cdot \frac{qf(e^z)}{qe^z} \cdot \frac{qe^z}{qz},$$

and $$\frac{q \log f(e^z)}{qf(e^z)} = \frac{1}{\log f(e^z)}, \quad \frac{qf(e^z)}{qe^z} = \frac{qy}{qx}, \quad \frac{qe^z}{qz} = z = \log x,$$

so that $$\frac{q \log f(e^z)}{qz} = \frac{\log x}{\log y} \cdot \frac{qy}{qx} = \frac{x \log x}{y \log y} \cdot \frac{dy}{dx},$$

and that this may equal unity, we must have $\frac{dy}{dx} = \frac{\log (y^y)}{\log (x^x)}$, which, therefore, is the condition that $\frac{ry}{rx} \neq \infty$.

This equation of condition may easily be solved, for, on separating the variables, we have

$$\frac{dy}{\log (y^y)} = \frac{dx}{\log (x^x)},$$

hence $$\int \frac{dy}{\log (y^y)} - \int \frac{dx}{\log (x^x)} = c, \text{ and since } \int \frac{dy}{\log (y^y)} = \log \log y,$$

the equation of condition reduces to

$$\log \log y - \log \log x = c.$$

This equation is transcendental in form only, for

$$\log \log y = c + \log \log x,$$
$$\log y = e^{c + \log \log x} = e^c . \log x,$$
$$y = e^{e^c \cdot \log x} = (e^{\log x})^{e^c} = x^{e^c} = x^n.$$

where $n = e^c$. We conclude, therefore, that $\frac{ry}{rx} = 1$ or ∞, according as $y =$ or is different from x^n, a fact which must render inadequate any calculus involving only these limiting processes.

13.—De Morgan's Extension of the Algebraic Processes.

It would appear from the foregoing articles that the algebraic processes do not admit of a generalization of the process of differentiation, that in fact the only possible extension of any consequence is quotientiation. This inference is

erroneous. The fault lies with the extension of the algebraic processes themselves. We shall see that $a^{\log b}$ as well as a^b may be looked upon as the $a \,)_2\, b$ of the ordinary algebra, and that when the former is chosen, a consistent calculus may be built up in which $\frac{d_2 y}{d_2 x}$ is the limiting process employed. In fact, by choosing properly the functions

$$\begin{aligned} &\omega_0(b) \quad , \quad \omega_1(b) \quad , \ldots . \ \omega_n(b) \ , \\ &\omega_{-1}(b), \quad \omega_{-2}(b), \ldots . \ \omega_{-n}(b), \end{aligned}$$

which enter into the definition equations of associated processes (art. 1), we shall arrive at a set of processes in which a $\frac{d_n y}{d_n x}$ exists for any two consecutive processes $)_{n-1}$ and $)_n$.

De Morgan* has shown how to extend the algebraic processes both forward and backward without violating the principle of the permanence of the formal laws of the addition and multiplication processes. *If, in the equations in art.* 1, *we put* $\omega_0(b) = b$, *the* $)_0$ *and* $)_1$ *processes can at once be identified with the ordinary arithmetic addition and multiplication. If, furthermore, we put*

$$\begin{aligned} \omega_1(b) &= b \,))\, r, \\ \omega_2(b) &= b \,))\, \overline{r}^{2}, \\ &\vdots \\ \omega_n(b) &= b \,))\, \overline{r}^{n}, \end{aligned}$$

where)) *represents any process such that*

$$a \,)_1\, b \,))\, r = a \,))\, r \,)_0\, (b \,))\, r),$$

and $$b \,))\, \overline{r}^{n} = b \,))\, r \,))\, r \,)) \ldots . \text{ to } b + 1 \text{ terms},$$

then $$a \,)_{n+1}\, b \,))\, r = a \,))\, r \,)_n\, (b \,))\, r). \qquad [\text{I}]$$

The proof of this proposition is easy. Let us assume that for some n

$$a \,)_n\, b \,))\, r = a \,))\, r \,)_{n-1}\, (b \,))\, r),$$

then, by definition,

$$a \,)_{n+1}\, b = a \,)_n\, a \,)_n \ldots . \text{ to } b \,))\, \overline{r}^{n} \text{ terms.}$$

---

* De Morgan's "Extension of the Algebraic Processes," by Christine Ladd, American Journal of Mathematics (1880), vol. 3.

hence

$$\begin{aligned}
a\,)_{n+1}\,b\,))\,r &= (a\,)_n\,a\,)_n \ \ldots\ \text{to}\ \overline{\overline{b\,))\,r}}^{n} - 1\ \text{terms})\,))\,r\,)_{n-1}(a\,))\,r),\\
&= (a\,)_n\,a\,)_n \ \ldots\ \text{to}\ \overline{\overline{b\,))\,r}}^{n} - 2\ \text{terms})\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1}(a\,))\,r),\\
&\ \ \vdots\\
&= a\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1} \ \ldots\ \text{to}\ \overline{b\,))\,r}^{n}\ \text{terms},\\
&= a\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1} \ \ldots\ \text{to}\ \overline{(b\,))\,r)\,))\,r}^{n-1}\ \text{terms},\\
&= a\,))\,r\,)_n(b\,))\,r)\ \text{by definition.}
\end{aligned}$$

The proposition holds, therefore, for any n if it holds for some particular n, but by hypothesis it holds when $n = 1$.

By a reapplication of [I] we have

$$\begin{aligned}
a\,)_n\,b\,))\,\overline{\overline{r}}^{n} &= a\,))\,r\,)_{n-1}\overline{(b\,))\,r)\,))\,r}^{n-1},\\
&= a\,))\,r\,)_{n-2}\overline{(b\,))\,r)\,))\,r}^{n-2},\\
&\ \ \vdots\\
&= a\,))\,r\,)_{n-p}\overline{(b\,))\,r)\,))\,r}^{n-p}, \qquad p \leqq n, \qquad \text{[II]}
\end{aligned}$$

when $p = n$, this becomes

$$a\,)_n\,b\,))\,\overline{\overline{r}}^{n} = a\,))\,\overline{r}^{n}\,)_0(b\,))\,\overline{r}^{n}). \qquad \text{[III]}$$

By definition, page 2,

$$a\,))\,r \asymp r = a,$$

so that we may put $a \asymp r = \overline{a\,))\,r}^{-1}$ without destroying the law of indices. Similarly,

$$a \asymp r \asymp r \asymp \underbrace{\text{to } n+1 \text{ terms}}_{-\,-\,-n} = a \asymp \overline{r}^{n} = \overline{a\,))\,r}^{-n}.$$

If, now, we take $\omega_{-n}(b) = b\,))\,r$, all of the processes in art. 1 come under the single formula

$$a\,)_n\,a\,)_n \ \ldots\ \text{to}\ \overline{b\,))\,r}^{n}\ \text{terms} = a\,)_{n+1}\,b,$$

when n may now be negative as well as positive. The fact that n is positive does not enter into the proof of formulæ [I], [II] and [III], so that these also hold when n is negative.

Formula [III] shows how any process, whether positive or negative, may be expressed in terms of the addition process, hence, the existence of a)) process

necessitates the existence of all the) processes. Now, $\log_b a$ satisfies the definition equation of the)) process for

$$\log(m \cdot n) = \log m + \log n,$$

hence *all the processes, both positive and negative exist and may be written out in terms of the addition process.*

Again, equation III, shows that the process $)_n$ is commutative; it has, therefore, but one inverse $\smile_n$, and since

$$\log(a \div b) = \log a - \log b,$$

the equation $$a \smile_1 b \,))\, r = a \,))\, r \smile_0 (b \,))\, r)$$

is satisfied, from which equations I, II, III follow with $)_n$ replaced by $\smile_n$. We have finally, in accordance with the definitions of article 1,

$$a \top_n b \,))\, \overline{\overline{r}}^{n} = a \,))\, \overline{\overline{r}}^{n} \top (b \,))\, \overline{\overline{r}}^{n}. \qquad [IV]$$

If the natural logarithmic process is the)) process employed,

$$a \,))\, r = \log a, \quad a \,))\, \overline{\overline{r}}^{-1} = e^a,$$

and [IV] becomes

$$a \top_n b = e^{\cdot^{\cdot^{(n)}}}, \qquad e^{(\log^n a \mp \log^n b)}$$

$$a \top_{-n} b = \log^n \left[e^{\cdot^{\cdot^{(n)}}} \mp e^{\cdot^{\cdot^{(n)}}} \right], \qquad e^a \quad e^b$$

according as the index of $\top$ is positive or negative, and $e^{\cdot^{\cdot^{(n)}}}$ signifies $e^{e^{\cdot^{\cdot}}}$ to n e's just as $\log^n$ signifies log log .... to n logs. If we write

$$e^{\cdot^{\cdot^{(n)}}} \qquad e^a$$

$$e^a = \log^{-1} a, \; e^{e^a} = \log^{-2} a, \text{ and generally, } e^{\cdot^{\cdot^{(n)}}} = \log^{-n} a,$$

$a\overline{\mathsf{T}}_n b$, and $a\mathsf{T}_{-n} b$ may be written

$$a\mathsf{T}_n b = \log^{-n}[\log^n a \mp \log^n b],$$
$$a\mathsf{T}_{-n} b = \log^n[\log^{-n} a \mp \log^{-n} b].$$

In particular, we have

$$\begin{aligned}
a\mathsf{T}_4 b &= \log^{-4}[\log^4 a \mp \log^4 b],\\
a\mathsf{T}_3 b &= \log^{-3}[\log^3 a \mp \log^3 b],\\
a\mathsf{T}_2 b &= \log^{-2}[\log^2 a \mp \log^2 b] = a^{1 \overset{+}{\times}{}^{\log b}},\\
a\mathsf{T}_1 b &= \log^{-1}[\log a \mp \log b] = a \overset{+}{\times} b,\\
a\mathsf{T}_0 b &= a \mp b, \qquad\qquad [\mathrm{A}]\\
a\mathsf{T}_{-1} b &= \log[\log^{-1} a \mp \log^{-1} b],\\
a\mathsf{T}_{-2} b &= \log^2[\log^{-2} a \mp \log^{-2} b],\\
a\mathsf{T}_{-3} b &= \log^3[\log^{-3} a \mp \log^{-3} b],\\
a\mathsf{T}_{-4} b &= \log^4[\log^{-4} a \mp \log^{-4} b],\\
&\text{etc.}
\end{aligned}$$

Finally, *each of the processes* T_n *admits a modulus.* To show this, we put in [III], $b = \overline{\overline{M_0))^{-n}} r}$, M_0 being the modulus of the process T_0. We get

$$a\mathsf{T}_n(\overline{\overline{M_0))^{-n}} r})\ \overline{))^{-n}} r = a\ \overline{))^{-n}} r\, \mathsf{T}(\overline{\overline{M_0))^{-n}} r}\ \overline{))^{-n}} r),$$
$$= a\ \overline{))^{-n}} r\, \mathsf{T} M_0 = a\ \overline{))^{-n}} r,$$

hence $a\mathsf{T}_n(\overline{\overline{M_0))^{-n}} r}) = a$, that is, $\overline{\overline{M_0))^{-n}} r} = M_n$.

But M_0 exists, being 0, hence

$$M_n = \log^{-n} 0, \quad M_{-n} = \log^n 0,$$

according as the process in question belongs to the positive or negative series In particular,

$$M_0 = 0, \quad M_1 = 1, \quad M_2 = e, \quad M_3 = e^e, \quad M_4 = e^{e^e}, \quad \text{etc.},$$
$$M_{-1} = -\infty, \quad M_{-2} = \log(-\infty)^* \quad M_{-3} = \log^2(-\infty), \quad \text{etc.}$$

14.—General Extension of the Differentiation Process.

The theorem last proven is of utmost importance. For, since every process $)_n$, as now defined, admits a modulus, it follows from theorem I, art. 2, that

$$\frac{d_n y}{d_n x} = M_{n-1} \frown_n M_{n-1}$$

* Here $\log(-\infty) = \log(-1) + \log\infty$, where $\log\infty = \lim_{x=\infty}[\log x]$, x being a positive real number.

is indeterminate, and it remains to show that this expression can be evaluated, i. e. that it has a distinctive determinate form in every case. This problem we will now consider. We begin with some special cases.

(a). $\frac{d_1 y}{d_1 x}$, $y = F(x)$.

Referring to the forms [A], art. 13, we have

$$\frac{d_1 y}{d_1 x} = \lim_{h = M_1} \left\{ F(x)_1 h) \frown_1 F(x) \frown_2 h \right\} = \lim_{h=1} \left\{ \log^{-2} \left[\log^2 \frac{F(xh)}{F(x)} - \log^2 h \right] \right\}$$

$$= \lim_{h=1} \left\{ \log^{-1} \frac{\log \frac{F(xh)}{F(x)}}{\log h} \right\} = \lim_{h=1} \left\{ \log^{-1} \log_h \frac{F(xh)}{F(x)} \right\}$$

$$= \log^{-1} \left\{ \lim_{h=1} \left[\log_h \frac{F(xh)}{F(x)} \right] \right\} = \log^{-1} \frac{qy}{qx}$$

$$= \log^{-1} \frac{d \log y}{d \log x} = e^{\frac{x}{y} \cdot \frac{dy}{dx}}.$$

(b). $\frac{d_2 y}{d_2 x}$.

$$\frac{d_2 y}{d_2 x} = \lim_{h = M_2} \{ F(x)_2 h) \frown_2 F(x) \frown_3 h \}$$

$$= \lim_{h=e} \left\{ \{ \log^{-3} \{ \log^3 \log^{-2} [\log^2 F(\log^{-2} \overline{\log^2 x + \log^2 h}) - \log^2 F(x)] - \log^3 h) \} \right\}.$$

Let us put $\log h = k$, $\log x = z$, then $k = 1$ when $h = e$, and

$$F(\log^{-2} \overline{\log^2 x + \log^2 h}) = F(e^{zk}),$$

$$\log^{-2} [\log^2 F(e^{zk}) - \log^2 (F(x)] = \log^{-1} \frac{\log F(e^{zk})}{\log F(e^z)},$$

$$\log^{-3} \left\{ \log^3 \log^{-1} \frac{\log F(e^{zk})}{\log F(e^z)} - \log^3 h \right\} = \log^{-2} \log_k \frac{\log F(e^{zk})}{\log F(e^z)},$$

hence

$$\frac{d_2 y}{d_2 x} = \log^{-2} \left\{ \lim_{k=1} \left[\log_k \frac{\log F(e^{zk})}{\log F(e^z)} \right] \right\} = \log^{-2} \frac{q \log F(e^z)}{qz}$$

$$= \log^{-2} \frac{q \log y}{q \log x} = \log^{-2} \frac{d \log^2 y}{d \log^2 x} = e^{e^{\left[\frac{\log x}{\log y} \cdot \frac{x}{y} \cdot \frac{dy}{dx} \right]}}.$$

(c). $\dfrac{d_{-2}y}{d_{-2}x}$.

$$\frac{d_{-2}y}{d_{-2}x} = \lim_{h=M_{-2}} \{F(x)_{-2}h) \frown_{-2} F(x) \frown_{-1} h\}$$

$$= \lim_{h=\log^2 0} \left\{ \log\{\log^{-1}\log^2[\log^{-2}F(\log^2\overline{\log^{-2}x+\log^{-2}h}) - \log^{-2}F(x)] - \log^{-1}h\} \right\}.$$

Put $\log^{-3}h = k$, $\log^{-3}x = z$, then $k = 1$ as $h = \log^2 0$, also

$$F(\log^2\overline{\log^{-2}x + \log^{-2}h}) = F(\log^3 zk),$$

$$\log^2[\log^{-2}F(\log^3 zk) - \log^{-2}F(x)] = \log^3\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\},$$

$$\log\left\{\log^{-1}\log^3\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\} - \log^{-1}h\right\} = \log^2\log_k\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\},$$

hence,

$$\frac{d_{-2}y}{d_{-2}x} = \log^2\left\{\lim_{k=1}\left[\log_k\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right]\right\} = \log^2\frac{q\,\log^{-3}F(\log^3 z)}{qz}$$

$$= \log^2\frac{q\,\log^{-3}y}{q\,\log^{-3}x} = \log^2\frac{d\,\log^{-2}y}{d\,\log^{-2}x}$$

$$= \log^2\left[\frac{e^{e^y}}{e^{e^x}}\cdot\frac{e^y}{e^x}\cdot\frac{dy}{dx}\right].$$

(d.) *The general case* $\dfrac{d_ny}{d_nx}$, *for any positive or negative index.*

Whether n is positive or negative, we have

$$\frac{d_ny}{d_nx} = \lim_{h=M_n}\{F(x)_n h) \frown_n F(x) \frown_{n+1} h\}$$

$$= \lim_{h=M_n}\left\{\log^{-(n+1)}\{\log^{n+1}\log^{-n}[\log^n F(\log^{-n}\overline{\log^n x + \log^n h}) - \log^n F(x)] - \log^{n+1}h\}\right\}$$

$$= \lim_{h=\log^{-n}0}\left\{\log^{-n}\frac{\log^n\log^{-(n-1)}\dfrac{\log^{n-1}F(\log^{-(n-1)}\overline{\log^{n-1}x.\,\log^{n-1}h})}{\log^{n-1}F(x)}}{\log^n h}\right\}.$$

If we now put $\log^{n-1}h = k$, $\log^{n-1}x = z$, then as h approaches the limit $\log^{-n}0$, k approaches the limit unity; also

$$\frac{\log^{n-1}F(\log^{-(n-1)}\overline{\log^{n-1}x\,.\,\log^{n-1}h})}{\log^{n-1}F(x)} = \frac{\log^{n-1}F(\log^{-(n-1)}\overline{zk})}{\log^{n-1}F(\log^{-(n-1)}z)},$$

so that

$$\frac{d_n y}{d_n x} = \lim_{k=1} \left\{ \log^{-n} \frac{\log \dfrac{\log^{n-1} F(\log^{-(n-1)} \overline{zk})}{\log^{n-1} F(\log^{-(n-1)} z)}}{\log k} \right\}$$

$$= \log^{-n} \left\{ \lim_{k-1} \left[\log_k \frac{\log^{n-1} F(\log^{-(n-1)} \overline{zk})}{\log^{n-1} F(\log^{-(n-1)} z)} \right] \right\}$$

$$= \log^{-n} \frac{q \log^{n-1} F(x)}{qz} = \log^{-n} \frac{q \log^{n-2} y}{q \log^{n-2} x}$$

$$= \log^{-n} \frac{d \log^n y}{d \log^n x}. \qquad [A]$$

When n is positive, this may be written

$$\frac{d_n y}{d_n x} = e^{\cdot^{\cdot^{\cdot^{(n)}}}} \qquad [B]$$

with the exponent $\left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \frac{\log^{n-2} x}{\log^{n-2} y} \cdots \frac{\log x}{\log y} \cdot \frac{x}{y} \cdot \frac{dy}{dx}\right]$.

When n is negative, say $n = -m$,

$$\frac{d_{\ m} y}{d_{-m} x} = \log^m \frac{d \log^{-m} y}{d \log^{-m} x} = \log^m \left\{ \begin{array}{cccc} e^y & e^y & & \\ \vdots & \vdots & & \\ (m) & (m-1) & & \\ \vdots & \vdots & & \\ \dfrac{e}{e^x} & \dfrac{e}{e^x} & \cdots & \dfrac{e^y}{e^x} \cdot \dfrac{dy}{dx} \\ \vdots & \vdots & & \\ (m) & (m-1) & & \\ \vdots & \vdots & & \\ e & e & & \end{array} \right\}. \qquad [C]$$

We shall call $\frac{d_n y}{d_n x}$ the *ratient* of the n^{th} order, so that the ratient of the zero order is the ordinary differential coefficient, the ratient of the first positive order is the exponential of the quotiential coefficient.

Formulæ [B] and [C] express the ratients of the n^{th} order in terms of differential coefficients, but it is possible likewise to express ratients of the n^{th} order in terms of the ratients of any lower or higher order. We have in fact

$$\frac{d_n y}{d_n x} = \log^{-n} \frac{d \log^n y}{d \log^n x} = \log^{-n} \left[\frac{d \log^n y}{d \log^{n-1} y} \cdot \frac{d \log^{n-1} y}{d \log^{n-1} x} \cdot \frac{d \log^{n-1} x}{d \log^n x} \right]$$

$$= \log^{-n} \left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \frac{d \log^{n-1} y}{d \log^{n-1} x} \right] = \log^{-n} \left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \log^{n-1} \log^{-(n-1)} \frac{d \log^{n-1} y}{d \log^{n-1} x} \right]$$

$$= \log^{-n} \left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \log^{n-1} \frac{d_{n-1} y}{d_{n-1} x} \right],$$

similarly,

$$\frac{d_{n-1} y}{d_{n-1} x} = \log^{-(n-1)} \left[\frac{\log^{n-2} x}{\log^{n-2} y} \cdot \log^{n-2} \frac{d_{n-2} y}{d_{n-2} x} \right],$$

so that

$$\frac{d_n y}{d_n x} = \log^{-n} \left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \frac{\log^{n-2} x}{\log^{n-2} y} \cdot \log^{n-2} \frac{d_{n-2} y}{d_{n-2} x} \right],$$

and, generally,

$$\frac{d_n y}{d_n x} = \log^{-n} \left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \frac{\log^{n-2} x}{\log^{n-2} y} \cdot \cdots \cdot \frac{\log^{n-r} x}{\log^{n-r} y} \cdot \log^{n-r} \frac{d_{n-r} y}{d_{n-r} x} \right]. \quad [D]$$

Again,

$$\frac{d_n y}{d_n x} = \log^{-n} \frac{d \log^n y}{d \log^n x} = \log^{-n} \left[\frac{d \log^n y}{d \log^{n+1} y} \cdot \frac{d \log^{n+1} y}{d \log^{n+1} x} \cdot \frac{d \log^{n+1} x}{d \log^n x} \right]$$

$$= \log^{-n} \left[\frac{\log^n y}{\log^n x} \cdot \frac{d \log^{n+1} y}{d \log^{n+1} x} \right] = \log^{-n} \left[\frac{\log^n y}{\log^n x} \cdot \log^{n+1} \log^{-(n+1)} \frac{d \log^{n+1} y}{d \log^{n+1} x} \right]$$

$$= \log^{-n} \left[\frac{\log^n y}{\log^n x} \cdot \log^{n+1} \frac{d_{n+1} y}{d_{n+1} x} \right],$$

and, generally,

$$\frac{d_n y}{d_n x} = \log^{-n} \left[\frac{\log^n y}{\log^n x} \cdot \frac{\log^{n+1} y}{\log^{n+1} x} \cdot \cdots \cdot \frac{\log^{n+r-1} y}{\log^{n+r-1} x} \cdot \log^{n+r} \frac{d_{n+r} y}{d_{n+r} x} \right]. \quad [E]$$

Successive ratients may be readily expressed in a compact form. For instance, whether n is positive or negative,

$$\frac{d_n^2 y}{d_n x^2} = \frac{d_n \left[\frac{d_n y}{d_n x} \right]}{d_n x} = \log^{-n} \frac{d \log^n \left[\frac{d_n y}{d_n x} \right]}{d \log^n x} = \log^{-n} \frac{d \left(\frac{d \log^n y}{d \log^n x} \right)}{d \log^n x}$$

$$= \log^{-n} \frac{d^2 \log^n y}{(d \log x)^2},$$

and hence, assuming that

$$\frac{d_n^{r-1}y}{d_n x^{r-1}} = \log^{-n}\frac{d^{r-1}\log^n y}{(d\log x)^{r-1}},$$

it follows by induction that generally

$$\frac{d_n^r y}{d_n x^r} = \log^{-n}\frac{d^r \log^n y}{(d\log x)^r}. \qquad [F]$$

15.—GENERAL RATIENTIATION FORMULÆ.

Let us now form the ratient of the m^{th} order of the $u \mathsf{T}_n v$, u and v being functions of x. We have from the preceding paragraph,

for $m \gtreqless n$,

$$\frac{d_m(u\mathsf{T}_n v)}{d_m x} = \log^{-m}\frac{d\log^m(u\mathsf{T}_n v)}{d\log^m x}$$

$$= \log^{-m}\left[\frac{d\log^m(u\mathsf{T}_n v)}{d\log^{m-1}(u\mathsf{T}_n v)}\cdot\frac{d\log^{m-1}(u\mathsf{T}_n v)}{d\log^{m-2}(u\mathsf{T}_n v)}\cdot\ldots\cdot\frac{d\log^{n+1}(u\mathsf{T}_n v)}{d\log^n(u\mathsf{T}_n v)}\cdot\frac{d\log^n(u\mathsf{T}_n v)}{d\log^m x}\right]$$

$$= \log^{-m}\left[\frac{1}{\log^{m-1}(u\mathsf{T}_n v)}\cdot\frac{1}{\log^{m-2}(u\mathsf{T}_n v)}\cdot\ldots\cdot\frac{1}{\log^n(u\mathsf{T}_n v)}\cdot\frac{d(\log^n u \mp \log^n v)}{d\log^m x}\right],$$

now

$$\frac{d\log^n u}{d\log^n x} = \frac{d\log^n u}{d\log^{n+1}u}\cdot\frac{d\log^{n+1}u}{d\log^{n+2}u}\cdot\ldots\cdot\frac{d\log^{m-1}u}{d\log^m u}\cdot\frac{d\log^m u}{d\log^m x}$$

$$= \log^n u\,.\log^{n+1}u\,.\ldots.\log^{m-1}u\,.\log^m\frac{d_m u}{d_m x},$$

so that finally

$$\frac{d_m(u\mathsf{T}_n v)}{d_m x} = \log^{-m}\frac{\log^n u\,.\log^{n+1}u\,.\ldots.\log^{m-1}u\,.\log^m\dfrac{d_m u}{d_m x} \mp \log^n v\,.\log^{n+1}v\,.\ldots.\log^{m-1}v\,.\log^m\dfrac{d_m v}{d_m x}}{\log^n(u\mathsf{T}_n v)\log^{n+1}(u\mathsf{T}_n v)\ldots.\log^{m-1}(u\mathsf{T}_n v)}; \qquad [A]$$

for $m \lessgtr n$,

$$\frac{d_m(u \mathsf{T}_n v)}{d_m x} = \log^{-m} \frac{d \log^m (u \mathsf{T}_n v)}{d \log^m x}$$

$$= \log^{-m} \left[\frac{d \log^m (u \mathsf{T}_n v)}{d \log^{m+1} (u \mathsf{T}_n v)} \cdot \frac{d \log^{m+1} (u \mathsf{T}_n v)}{d \log^{m+2} (u \mathsf{T}_n v)} \cdots \frac{d \log^{n-1} (u \mathsf{T}_n v)}{d \log^n (u \mathsf{T}_n v)} \cdot \frac{d \log^n (u \mathsf{T}_n v)}{d \log^m x} \right]$$

$$= \log^{-m} \left[\log^m (u \mathsf{T}_n v) . \log^{m+1} (u \mathsf{T}_n v) \cdots \log^{n-1} (u \mathsf{T}_n v) . \frac{d (\log^n u \mp \log^n v)}{d \log^m x} \right].$$

and since

$$\frac{d \log^n u}{d \log^m x} = \frac{d \log^n u}{d \log^{n-1} u} \cdot \frac{d \log^{n-1} u}{d \log^{n-2} u} \cdots \frac{d \log^{m+1} u}{d \log^m u} \cdot \frac{d \log^m u}{d \log^m x}$$

$$= \frac{1}{\log^{n-1} u} \cdot \frac{1}{\log^{n-2} u} \cdots \frac{1}{\log^m u} \cdot \log^m \frac{d_m u}{d_m x},$$

we have

$$\frac{d_m(u \mathsf{T}_n v)}{d_m x} = \log^{-m} \frac{(\log^{n-1} u . \log^{n-2} u \cdots \log^n u)^{-1} \log^m \dfrac{d_m u}{d_m x} \mp (\log^{n-1} v . \log^{n-2} v \cdots \log^m v)^{-1} \log^m \dfrac{d_m v}{d_m x}}{[\log^{n-1} (u \mathsf{T}_n v) . \log^{n-2} (u \mathsf{T}_n v) \cdots \log^m (u \mathsf{T}_n v)]^{-1}}. \quad [\mathrm{B}]$$

Formulæ [A] and [B] enable us to express the ratient of two functions combined by any process, as exponentials or logarithms of sums of multiples of the ratients of the functions taken separately. The extension of the formulæ to the distribution of dm over u, v, w, etc., in $\frac{d_m (u \mathsf{T}_n v \mathsf{T}_n w \mathsf{T}_n, \text{ etc.}}{d_m x}$ is obvious.

When $m = 0$, [A] and [B] become respectively

$$\frac{d(u \mathsf{T}_n v)}{dx} = \frac{\log^n u . \log^{n+1} u \ldots \log^{-1} u . \dfrac{du}{dx} \mp \log^n v . \log^{n+1} v \ldots \log^{-1} v . \dfrac{dv}{dx}}{\log^n (u \mathsf{T}_n v) \log^{n+1} (u \mathsf{T}_n v) \cdots \log^{-1} (u \mathsf{T}_n v)}, \quad n<0,$$

$$\frac{d(u \mathsf{T}_n v)}{dx} = (u \mathsf{T}_n v) \log (u \mathsf{T}_n v) \cdots \log^{n-1} (u \mathsf{T}_n v)$$

$$\times \left[\frac{1}{u} \cdot \frac{1}{\log u} \cdots \frac{1}{\log^{n-1} u} \frac{du}{dx} \mp \frac{1}{v} \cdot \frac{1}{\log v} \cdots \frac{1}{\log^{n-1} v} \cdot \frac{dv}{dx} \right], \quad n>0,$$

which are the formulæ for the *differential coefficient of two functions combined by the n^{th} process.*

When $n = 0$, [A] and [B] yield

$$\frac{d_m(u \mp v)}{d_m x}$$

$$= \log^{-m} \frac{u \log u \ldots \log^{m-1} u \,.\, \log^m \frac{d_m u}{d_m x} \mp v \log v \ldots \log^{m-1} v \,.\, \log^m \frac{d_m v}{d_m x}}{(u+v)\log(u+v) \ldots \log^{m-1}(u+v)} \quad m > 0,$$

$$\frac{d_m(u \mp v)}{d_m x} = \log^{-m} \frac{(\log^{-1} u \,.\, \log^{-2} u \ldots \log^m u)^{-1} \log_m \frac{d_n u}{d_n x} \mp (\log^{-1} v \,.\, \log^{-2} v \ldots \log^m v)^{-1} \log^m \frac{d_m v}{d_m x}}{\left(\log^{-1}(u+v)\, \log^{-2}(u+v) \ldots \log^m (u+v)\right)^{-1}}, \quad m < 0.$$

formulæ which express *the ratient of a sum of functions in terms of the ratients of the functions taken separately.*

Putting $n = 1$, we get corresponding expressions for the *ratient of a product.*

When $n = 1$, $m = 0$, we get from [B] the *differential coefficient of a product.*

$n = 2$, $m = 0$ gives, after replacing $\log v$ by v in the final result, and taking the lower sign,

$$\frac{du^v}{dx} = vu^{v-1} \frac{du}{dx} + \log u \,.\, u^v \frac{dv}{dx}.$$

These special cases suffice to indicate the extreme generality of formulæ [A] and [B].

Finally, it is not difficult to show that for $m = n$ [A] or [B] leads to

$$\frac{d_n(u \mathbin{\top}_n v)}{d_n x} = \frac{d_n u}{d_n x} \mathbin{\top}_n \frac{d_n v}{d_n x}, \qquad [C]$$

and for $n = m + 1$, [B] becomes, for the lower sign,

$$\frac{d_m(u|_{m+1} v)}{d_m x} = \frac{d_m u}{d_m x} |_{m+1} v |_m \left(\frac{d_m v}{d_m x} |_{m+1} u\right), \qquad [D]$$

and for the upper sign,

$$\frac{d_m(u -_{m+1} v)}{d_m x} = \frac{d_m u}{d_m x} -_{m+1} v -_m \left(\frac{d_m v}{d_m x} |_{m+1} u -_{m+1} \overline{v |_{m+2} \log^{-(n+1)}} 2\right). \qquad [D']$$

[C], [D] and [D′] are generalizations respectively of

$$\frac{d(u \mp v)}{dx} = \frac{du}{dx} \mp \frac{dv}{dx}, \quad \frac{d(u \,.\, v)}{dx} = \frac{du}{dx} \,.\, v + \frac{dv}{dx} \,.\, u,$$

and $$\frac{d(u \div v)}{dx} = \frac{du}{dx} \div v - \frac{dv}{dx} \cdot u \div v^2.$$

The formulæ

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{dy}{dx} = 1 \div \frac{dx}{dy},$$

assumes the generalized forms

$$\frac{\partial_n^1 u}{\partial_n x \partial_n y} = \frac{\partial_n^2 u}{\partial_n y \partial_n x}, \quad \frac{d_n y}{d_n x} = M_{n+1} -_{n+1} \frac{d_n x}{d_n y},$$

and in fact every theorem or formula in the ordinary calculus has its analogue in the calculus of ratients.

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Simple Pairs of Parallel W-Surfaces.

By Henry Dallas Thompson.

Formulas between the Elements of Parallel Surfaces.

Let x, y, z be the coordinates of any point M on a given surface S; and let X, Y, Z; E, F, G; D, D', D''; e, f, g; H, K; r_1, r_2, etc., have the usual meaning on this surface; r_1, r_2 being measured *toward* the point M. [B.-L., p. 98]. (For convenience use will be made of the formulas of Bianchi, "Differentialgeometrie," Deutsche Uebersetzung von Lukat, Leipzig, 1899, and such references will be indicated by B.-L.). Let l be a *constant* which gives the length (positive or negative) from some point M_0 on the normal at the point M; then, if x', y', z' be the running coordinates,

$$(x'-x)X+(y'-y)Y+(z'-z)Z+l=0 \tag{1}$$

is the equation to the plane parallel to the tangent plane at M, and at the distance $-l$ from M. To find the envelope S' of the plane (1), that is, the "parallel surface" to S with the constant distance l between the parallel tangent planes, differentiate (1) with regard to u and v respectively, which gives

$$\begin{aligned}(x'-x)\frac{\partial X}{\partial u}+(y'-y)\frac{\partial Y}{\partial u}+(z'-z)\frac{\partial Z}{\partial u}=0,\\ (x'-x)\frac{\partial X}{\partial v}+(y'-y)\frac{\partial Y}{\partial v}+(z'-z)\frac{\partial Z}{\partial v}=0.\end{aligned} \tag{2}$$

The solution of the equations (1) and (2) for $(x'-x)$, $(y'-y)$, $(z'-z)$ is (using B.-L., p. 128, mid.),

$$x'-x=-lX,\quad y'-y=-lY,\quad z'-z=-lZ. \tag{3}$$

Thus the point M' is the point M_0, and the parallel surface S' is the dilated surface S_0, of which Bonnet speaks in N. Annal. (1853), pp. 433–8.

Let x_0, y_0, z_0; X, Y, Z; E_0, F_0, G_0, etc., be the elements of the point M_0 of the surface S_0 corresponding to M of the surface S. Differentiation gives

$$\begin{aligned} \frac{\partial x_0}{\partial u} &= \frac{\partial x}{\partial u} - l\frac{\partial X}{\partial u}, \quad \frac{\partial y_0}{\partial u} = \frac{\partial y}{\partial u} - l\frac{\partial Y}{\partial u}, \quad \frac{\partial z_0}{\partial u} = \frac{\partial z}{\partial u} - l\frac{\partial Z}{\partial u}, \\ \frac{\partial x_0}{\partial v} &= \frac{\partial x}{\partial v} - l\frac{\partial X}{\partial v}, \quad \frac{\partial y_0}{\partial v} = \frac{\partial y}{\partial v} - l\frac{\partial Y}{\partial v}, \quad \frac{\partial z_0}{\partial v} = \frac{\partial z}{\partial v} - l\frac{\partial Z}{\partial v}. \end{aligned} \tag{4}$$

and thence follow directly from the definitions,

$$\begin{aligned} &E_0 = E + 2lD + l^2e, \quad F_0 = F + 2lD' + l^2f, \quad G_0 = G + 2lD'' + l^2g, \\ &D_0 = D + le \qquad\quad , \quad D_0' = D' + lf \qquad\quad , \quad D_0'' = D'' + lg \qquad . \end{aligned} \tag{5}$$

Setting the values of e, f, g from

$$e + KE + HD = 0, \quad f + KF + HD' = 0, \quad g + KG + HD'' = 0 \quad \text{[B.-L., p. 119]}$$

in (5), these formulas become

$$\begin{aligned} E_0 &= E + 2lD - l^2(EK + DH), \quad D_0 = D - l(EK + DH), \\ F_0 &= F + 2lD' - l^2(FK + D'H), \quad D_0' = D' - l(FK + D'H), \\ G_0 &= G + 2lD'' - l^2(GK + D''H), \quad D_0'' = D'' - l(GK + D''H). \end{aligned} \tag{5'}$$

Using the values of H and K [B.-L., p. 105 (18)]:

$$K = \frac{DD'' - D'^2}{EG - F^2}, \quad H = \frac{2FD' - ED'' - GD}{EG - F^2},$$

direct calculation from the formulas (5′) gives:

$$\begin{aligned} E_0G_0 - F_0^2 &= (EG - F^2) + 2l(ED'' + GD - 2FD') \\ &\quad + l^2[4(DD'' - D'^2) - K(EG + GE - 2F^2) - H(ED'' + GD - 2FD')] \\ &\quad - 2l^3[K(GD + ED'' - 2FD') + 2H(DD'' - D'^2)] \\ &\quad + l^4[K^2(EG - F^2) + HK(ED'' + GD - 2FD') + H^2(DD'' - D'^2)] \\ &= (EG - F^2)[1 - 2lH + l^2(4K - 2K + H^2) + 2l^3(KH - 2HK) \\ &\quad + l^4(K^2 - H^2K + H^2K)] \\ &= (EG - F^2)(1 - lH + l^2K)^2, \end{aligned}$$

$$
\begin{aligned}
D_0D_0'' - D_0'^2 &= (DD'' - D'^2) - l\,[K(ED'' + GD - 2FD') + 2H(DD'' - D'^2)] \\
&\quad + l^2\,[K^2(EG - F^2) + HK(ED'' + GD - 2FD') + H^2(DD'' - D'^2)] \\
&= (EG - F^2)[K + l(KH - 2HK) + l^2(K^2 - H^2K + H^2K)] \\
&= (EG - F^2)(1 - lH + l^2K)\,K,
\end{aligned}
$$

$$
\begin{aligned}
2F_0D_0' - E_0D_0'' - G_0D_0 &= (2FD' - ED'' - GD) \\
&\quad - l\,[-4(D'^2 - DD'') + 2K(F^2 - EG) + H(2FD' - ED'' - GD)] \\
&\quad - l^2\,[K(6FD' - 3ED'' - 3GD) + H(6D'^2 - 3DD'' - 3D''D)] \\
&\quad + 2l^3\,[K^2(F^2 - EG) + HK(2FD' - ED'' - GD) + H^2(D'^2 - DD'')] \\
&= (EG - F^2)[H - l(4K - 2K + H^2) - l^2(3KH - 6HK) \\
&\qquad\qquad + 2l^3(-K^3 + H^2K - H^2K)] \\
&= (EG - F^2)(1 - lH + l^2K)(H - 2lK).
\end{aligned}
$$

Combinations of these three formulas give:

$$K_0 = \frac{K}{1 - lH + l^2K}, \quad H_0 = \frac{H - 2lK}{1 - lH + l^2K}; \tag{6}$$

whence,

$$1 + lH_0 + l^2K_0 = \frac{1}{1 - lH + l^2K},$$

and, therefore, finally also,

$$K = \frac{K_0}{1 + lH_0 + l^2K_0}, \quad H = \frac{H_0 + 2lK_0}{1 + lH_0 + l^2K_0}. \tag{7}$$

The value of K_0 in (6) has been given by Craig, Jour. für Math., 94 (1883), pp. 162–170.

To the lines of curvature on S correspond the lines of curvature on S_0; for, from (5), if $F = 0$, $D' = 0$ and $f = 0$, then $F_0 = 0$ and $D_0' = 0$. And the corresponding lines of curvature on the two surfaces have parallel tangents, for the direction cosines are in proportion, viz. [B.-L., p. 63]:

$$\cos(u, x) = \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \quad \cos(u, y) = \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v}, \quad \cos(u, z) = \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v}, \tag{8}$$

$$\cos(u, x_0) = \frac{1}{\sqrt{G_0}}\frac{\partial x_0}{\partial v}, \quad \cos(u, y_0) = \frac{1}{\sqrt{G_0}}\frac{\partial y_0}{\partial v}, \quad \cos(u, z_0) = \frac{1}{\sqrt{G_0}}\frac{\partial z_0}{\partial v}, \tag{9}$$

and setting in the values from (4) and using the formulas B.-L., p. 102 (13), the

cosines of (9) become:

$$\frac{1}{\sqrt{G_0}}\frac{\partial x}{\partial v}\left(1-\frac{l}{r_1}\right),\quad \frac{1}{\sqrt{G_0}}\frac{\partial y}{\partial v}\left(1-\frac{l}{r_1}\right),\quad \frac{1}{\sqrt{G_0}}\frac{\partial z}{\partial v}\left(1-\frac{l}{r_1}\right),$$

and are proportional to the cosines of (8). And similarly for the other line of curvature.

Four Triples of Parallel W-Surfaces.

The equations (6) lead to pairs of simple parallel W-surfaces when $K_0 = \pm K$ or $H_0 = \pm H$.

Case a. If $K_0 = K$, the first equation of (6) gives $l = H/K = r_1 + r_2$. This equation, $l = r_1 + r_2$, may be called the Weingarten equation for this surface. In this case, $H_0 = -H$. (The solution $l = 0$ corresponds to the original surface.) Then $r_1 = l - r_2 = -r_{02}$, and $r_2 = l - r_1 = -r_{01}$. The surface S' at the distance $-\frac{1}{2}l$ from S, that is, the surface midway between S and S_0 will have, from (6), the mean curvature $H' = 0$; and then from (7),

$$H = \frac{lK'}{1+\frac{1}{4}l^2K'},\quad K = \frac{K'}{1+\frac{1}{4}l^2K'}.$$

This may be expressed: The two parallel surfaces on the two sides of a minimal surface at any constant distances $\pm\frac{1}{2}l$, have at corresponding points their principal curvatures equal, and the mean curvatures equal but with opposite signs, and each radius of principal curvature equal to the negative of the non-corresponding radius of principal curvature of the other surface. A figure showing these relations may be obtained by taking two equidistant surfaces parallel to the catenoid.

Case b. *If* $K_0 = -K$, the first of (6) gives $K = 0$, or

$$l^2K - lH + 2 = 0. \tag{10}$$

Then the original W-surface being given by the Weingarten equation (10), the mid-surface S' at the distance $-\frac{1}{2}l$ will have, from (6) and (10), $K' = -4/l^2$. And, from (7), the two parallel surfaces at the constant distances $\pm\frac{1}{2}l$ from a surface with constant principal curvature $-4/l^2$, at corresponding points, have their principal curvatures equal but of opposite signs. A figure showing these

relations may be obtained by taking surfaces parallel to a surface of revolution with constant negative curvature.

Case c. If $H_0 = H$, the second of (6) gives the Weingarten equation

$$l = H/K - 2/H = (r_1^2 + r_2^2)/(r_1 + r_2), \tag{11}$$

and for the second surface, $-l = (r_{02}^2 + r_{01}^2)/(r_{02} + r_{01})$. To get the mid-surface S', the combination of the two formulas (7) gives: $H'/K' = H/K - l$. And (11) then becomes, using the second of (7),

$$l = H'/K' + l - 2(1 + \tfrac{1}{2} lH' + \tfrac{1}{4} l^2 K')/(H' + lK'),$$

or
$$H'^2 + lH'K' = 2K' + lH'K' + \tfrac{1}{2} l^2 K'^2,$$

or finally
$$\tfrac{1}{2} l^2 = (H'^2 - 2K')/K'^2 = r_1'^2 + r_2'^2.$$

Case d. If $H_0 = -H$, the second of (6) gives $(lK - H)(lH - 2) = 0$, and the first factor equal to zero has been considered under case *a*. The consideration of the second factor gives the Weingarten equation $l = 2/H = 2r_1r_2/(r_1+r_2)$. This surface, H equal to a constant, has been studied by Bonnet,* who has shown, as follows also at once from (6), that, for the mid-surface S', $K' = 4/l^2 = H^2$. The two cases *b* and *d* give one pair of *real* surfaces parallel to and equidistant from *every* real surface with constant measure of curvature. Several well known theorems, some in an extended form, follow from (5) and the formulas immediately following. Every pair of orthogonal lines on one surface will correspond to a pair of orthogonal lines on a parallel surface only when the surfaces have constant mean curvature equal but with opposite signs. For, if $F_0 = 0$ and $F = 0$, with D' not zero and f not zero, then $2D' + lf = 0$ and $0 = f + HD'$; hence, $l = 2/H$. The asymptotic lines on one surface correspond to a pair of conjugate lines on the parallel surface only when the surfaces have constant mean curvature, equal but with opposite signs. For, from (5),

$$D_0D'' + D_0''D - 2D_0'D' = 2(DD'' - D'^2) + l(eD'' + gD - 2D'f).$$

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* Bonnet, Nouv. Ann., 12 (1853), pp. 433–8. See also: Jellett, Jour. de Math., 18 (1853), pp. 163–7. Bour, Éc. Polyt., 39 (1862), pp. 109 f. Bonnet, Éc. Polyt., 42 (1867), p. 77. Simon, Diss., Halle, 1876. Willgrad, Diss., Göttingen, 1883. Chini, Gior. di Mat., 27 (1889), pp. 107–123. Darboux, "Surfaces," vol. II, pp. 243 ff.; vol. III, pp. 375 f. Vivanti, Lomb. Rend. (2), 27 (1895), p. 699.

And the left-hand side of this equation equated to zero is the necessary and sufficient condition for the correspondence of the asymptotic with a pair of conjugate lines. [B.-L., p. 293.] But the right-hand side equated to zero gives $l = 2/H$. [B.-L., p. 124, (8)]. A pair of orthogonal lines on one surface corresponds to a pair of conjugate lines on a parallel surface only when the first surface has constant mean curvature and the second surface is the parallel mid-surface with constant principal curvature. For, $F = 0$ and $D_0' = 0$ in (5) and the following formulas give : $0 = D' + lf$, $f + HD' = 0$ or $l = 1/H$.

Pairs of Anharmonic Parallel Surfaces.

The constant distance M_0M in the case d is $2/H$. If r_1 be M_1M and r_2 be M_2M, it is seen at once that here M_0 is the fourth harmonic to the points M_1, M, M_2. The question at once arises, for what surface will the line M_0M be a constant, when M_0 is the fourth *anharmonic* to the points M_1, M, M_2? Here, a and b being constants,

$$r_1 : r_2 = aM_1M_0 : -bM_2M_0.$$

Setting in $M_1M_0 = r_1 - M_0M$, $M_2M_0 = r_2 - M_0M$, this reduces to $M_0M(ar_2 + br_1) = (a + b)\, r_1r_2$, or finally :

$$M_0M = \frac{a+b}{\dfrac{a}{r_1} + \dfrac{b}{r_2}}.$$

In order that M_0M shall be a constant, it is necessary and sufficient that the denominator be constant, say c; that is, $M_0M = (a + b)/c$, and the Weingarten equation is the most general linear relation between the reciprocals of the principal radii of curvature,

$$\frac{a}{r_1} + \frac{b}{r_2} = c. \tag{12}$$

Let this surface be referred to its lines of curvature; the parallel surface at the distance $-M_0M$ or $-l$, will have from (5) and B.-L., p. 102, (14) and B.-L., p. 235, formulas between (3) and (4),

$$E_0 = E(1 - l/r_2)^2, \quad G_0 = G\ (1 - l/r_1)^2,$$
$$D_0 = D(1 - l/r_2), \quad D_0'' = D''(1 - l/r_1),$$

and then $r_{01} = r_1(1 - l/r_1)$ and $r_{02} = r_2(1 - l/r_2)$, or

$$r_1 = r_{01} + l, \quad r_2 = r_{02} + l. \tag{13}$$

These values set in (12) give:

$$a(r_{02} + l) + b(r_{01} + l) = c(r_{01} + l)(r_{02} + l),$$

or
$$(a - cl)r_{02} + (b - cl)r_{01} - cr_{01}r_{02} = l(cl - a - b).$$

But setting in the value $l = (a + b)/c$ and dividing by $-r_{01}r_{02}$, this becomes the anharmonic surface, with the linear Weingarten equation:

$$\frac{b}{r_{01}} + \frac{a}{r_{02}} = -c. \tag{14}$$

The relation between the principal radii of curvature of all surfaces parallel to the original surface satisfying the equation (12) may be found. To this end, the values (13) set in (12) give again:

$$a(l + r_{02}) + b(l + r_{01}) = c(l + r_{01})(l + r_{02})$$

or
$$\frac{\{(a+b) - cl\}\, l}{r_{01}r_{02}} + \frac{a - cl}{r_{01}} + \frac{b - cl}{r_{02}} - c = 0, \tag{15}$$

From this equation it follows: There is one and only one other surface parallel to the surface satisfying (12) which has a linear relation between the reciprocals of the principal radii of curvature, and that is the surface at the distance l given by $(a + b) - cl = 0$, or $l = (a + b)/c$; and for this value of l, $a - cl = -b$, and $b - cl = -a$, so that (15) in this case becomes:

$$\frac{b}{r_{01}} + \frac{a}{r_{02}} + c = 0. \tag{14}$$

It is further seen from (15), that no parallel surface to (12) has a constant K_0 or a linear relation between H_0 and K_0 (unless $a = b$, where (12) reduces to case d above).

The surface midway between (12) and (14) is found by setting $l = (a+b)/2c$ in the formula (15), and this gives:

$$\frac{(a+b)^2}{4c} \cdot \frac{1}{r_1' r_2'} + \frac{a - b}{2}\left(\frac{1}{r_1'} - \frac{1}{r_2'}\right) - c = 0. \tag{16}$$

And the pairs of parallel surfaces on each side of (16) at the distances $\pm l'$ respectively, satisfy the equations:

$$\frac{1}{c}\left\{\frac{(a+b)^2}{4}-c^2l'^2\right\}\frac{1}{r_1r_2}+\frac{a-b}{2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\mp cl'\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-c=0. \quad (17)$$

It will have been noticed that the Weingarten equations for the *pairs* of parallel surfaces (not mid-surfaces) in cases a, b, c, and d, and also in the anharmonic case (17), can be derived one from the other by writing $-r_{01}$ and $-r_{02}$ for r_2 and r_1 respectively.

PRINCETON, N. J., *October* 7, 1901.

*On Systems of Linear Differential Equations of the First Order.**

By Maxime Bôcher.

It is my object in this paper to consider the system of n differential equations:

$$\frac{dy_i}{dx} = \sum_{j=1}^{j=n} \alpha_{ij} y_j + \beta_i, \qquad (i = 1, 2, \ldots . n), \qquad (1)$$

where the coefficients α and β are functions, not necessarily analytic, of the real variable x which, throughout the interval

$$(I) \qquad a \leqq x \leqq b,$$

satisfy the following condition:

A. *The functions α_{ij} and β_i shall have at most a finite number of discontinuities in (I), and the integrals*

$$\int |\alpha_{ij}| dx, \quad \int \beta_i dx$$

shall converge, when extended over any portion of (I).

We add the definition:

B. *A set of n functions of x $(y_1, y_2, \ldots . y_n)$ are said to form a solution of (1) when, and only when, they are all of them continuous at every point of (I), and, except at most at a finite number of points of this interval, have derivatives which satisfy (1).*

The first theorem we wish to prove is the following:

I. *Any set of n constants $(c_1, c_2, \ldots . c_n)$, and any point c of (I) being given, there exists one, and only one, solution $(y_1, y_2, \ldots . y_n)$ of (1) which satisfies the initial conditions:*

$$y_i(c) = c_i, \qquad (i = 1, 2, \ldots . n). \qquad (2)$$

* This paper was read before the American Mathematical Society, February 22, 1902.

To prove that at least one such solution exists, we use the method of successive approximations.* For this purpose consider the series

$$y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \ldots . \qquad (i = 1, 2, \ldots . n), \qquad (3)$$

where

$$y_i^{(0)} = c_i + \int_c^x \beta_i \, dx,$$

and, when $k > 0$,

$$y_i^{(k)} = \int_c^x \Big(\sum_{j=1}^{j=n} a_{ij} y_j^{(k-1)} \Big) \, dx. \qquad (i = 1, 2, \ldots . n), \qquad (4)$$

It is clear that the integrals here written all converge, and represent functions which are continuous throughout (I).

In order to prove the convergence of the series (3), let M be a positive constant such that

$$|y_i^{(0)}| \leqq M, \qquad (i = 1, 2, \ldots . n), \qquad (5)$$

and let $\phi(x)$ be a function which has at most a finite number of discontinuities in (I), for which the integral

$$\int \phi(x) \, dx$$

converges when extended over any portion of (I), and such that

$$|a_{ij}| \leqq \phi(x), \dagger \qquad (i, j = 1, 2, \ldots . n). \qquad (6)$$

We can now establish the fundamental inequalities:

$$|y_i^{(k)}| \leqq M \frac{n^k \left| \int_c^x \phi(x) \, dx \right|^k}{k!} \qquad \begin{pmatrix} i = 1, 2, \ldots . n \\ k = 0, 1, \ldots . \end{pmatrix}. \qquad (7)$$

These inequalities reduce to (5) when $k = 0$. We shall therefore have established their truth by the method of mathematical induction if, assuming (7) to

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* This method was used for this purpose for the first time by Peano; cf. Mathematische Annalen, vol. 32 (1888), p. 450. Peano, however, restricts himself not merely to homogeneous equations (i. e. assumes $\beta_1 \equiv \beta_2 \equiv \ldots \equiv \beta_n \equiv 0$), but also restricts the coefficients a_{ij} to be continuous throughout (I).

† That such a function exists will be seen from the example:

$$\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} |a_{ij}|.$$

hold when $k \leqq k_1$, we can prove that it holds when $k = k_1 + 1$. Now we have:

$$|y_i^{(k_1+1)}| \leqq \int_c^x \Big(\sum_{j=1}^{j=n} |a_{ij}| \cdot |y_j^{(k_1)}|\Big)|dx| \leqq M\frac{n^{k_1+1}}{k_1!}\Big|\int_c^x \phi(x)\Big(\int_c^x \phi(x)\,dx\Big)^{k_1} dx\Big|. \quad (8)$$

By integrating by parts we find

$$\int_c^x \phi(x)\Big(\int_c^x \phi(x)\,dx\Big)^{k_1} dx = \frac{\Big(\int_c^x \phi(x)\,dx\Big)^{k_1+1}}{k_1+1}. \quad (9)$$

The substitution of (9) in (8) gives us (7) for the case $k = k_1 + 1$.

If, now, we write

$$N = \int_a^b \phi(x)\,dx, \quad (10)$$

we deduce at once from (7) the further inequality:

$$|y_i^{(k)}| \leqq M\frac{(nN)^k}{k!}, \qquad \begin{pmatrix} i = 1, 2, \ldots n \\ k = 0, 1, \ldots \end{pmatrix}. \quad (11)$$

If, then, we compare (3) with the convergent series of positive constant terms,

$$\sum_{k=0}^{k=\infty} M\frac{(nN)^k}{k!}, \quad (12)$$

we see that the series (3) are absolutely and uniformly convergent throughout (I). Writing

$$y_i = y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \ldots, \qquad (i = 1, 2, \ldots n) \quad (13)$$

we see that the functions $(y_1, y_2, \ldots y_n)$ thus defined are continuous throughout (I). Moreover, since, when $x = c$, all the terms except the first of these series vanish, while $y_i^{(0)}(c) = c_i$, it follows that $y_i(c) = c_i$.

In order to prove that $(y_1, \ldots y_n)$ form a solution of (1) according to the definition (B), it will then be sufficient if we can prove that, x_0 being any point where the coefficients α_{ij} and β_i are continuous, $(y_1, \ldots y_n)$ have derivatives at x_0 which satisfy (1).

For this purpose let us surround x_0 by a short interval (J) throughout which the coefficients α_{ij} and β_i are continuous. We have:

$$\sum_{j=1}^{j=n} \alpha_{ij} y_j + \beta_i = \beta_i + \alpha_{i_1}\sum_{k=0}^{k=\infty} y_1^{(k)} + \alpha_{i2}\sum_{k=0}^{k=\infty} y_2^{(k)} + \ldots + \alpha_{in}\sum_{k=0}^{k=\infty} y_n^{(k)}$$

$$= \beta_i + \sum_{j=1}^{j=n} \alpha_{ij} y_j^{(0)} + \sum_{j=1}^{j=n} \alpha_{ij} y_j^{(1)} + \ldots \qquad (i = 1, 2, \ldots n).$$

This last series clearly converges uniformly throughout (J), since the series (13) do so. It is, however, precisely the series which we should obtain by differentiating (13) term by term. Accordingly y_i has a derivative throughout (J), and this derivative is represented by the series last written. Moreover, the formula just obtained shows that this derivative satisfies (1). Thus our proof is complete.

It remains merely to prove that two solutions $(y_1, \ldots . y_n)$ and $(\bar{y}_1, \ldots . \bar{y}_n)$ of (1) which satisfy the conditions

$$y_i(c) = \bar{y}_i(c) \qquad (i = 1, 2, \ldots . n)$$

must be identically equal. For this purpose we notice that if

$$\eta_i = y_i - \bar{y}_i, \qquad (i = 1, 2, \ldots . n),$$

then $(\eta_1, \eta_2, \ldots . \eta_n)$ is a solution of the system of homogeneous linear equations

$$\frac{d\eta_i}{dx} = \sum_{j=1}^{j=n} \alpha_{ij}\eta_j, \qquad (i = 1, 2, \ldots . n). \qquad (14)$$

Our theorem will, therefore, be established if we can prove the following special case of it:

If $(\eta_1, \eta_2, \ldots . \eta_n)$ *is a solution of* (14), *and if*

$$\eta_i(c) = 0, \qquad (i = 1, 2, \ldots . n),$$

then

$$\eta_i(x) \equiv 0, \qquad (i = 1, 2, \ldots . n).$$

If possible, let h be a point in (I), where all the functions η_i do not vanish. Then we can, by the portion of theorem I already proved, find $n - 1$ solutions of (14),

$$(\eta_1^{(i)}, \eta_2^{(i)}, \ldots . \eta_n^{(i)}) \qquad (i = 1, 2, \ldots . n - 1)$$

such that the determinant

$$D(x) = \begin{vmatrix} \eta_1 & \eta_1^{(1)} & \ldots . & \eta_1^{(n-1)} \\ \eta_2 & \eta_2^{(1)} & \ldots . & \eta_2^{(n-1)} \\ \vdots & \vdots & & \\ \eta_n & \eta_n^{(1)} & \ldots . & \eta_n^{(n-1)} \end{vmatrix}$$

does not vanish when $x = h$.*

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* If, for instance $\eta_n(h) \neq 0$, we may determine these solutions by the conditions:

$$\eta_1^{(1)}(h) = \eta_2^{(2)}(h) = \ldots . = \eta_{n-1}^{(n-1)}(h) = 1,$$

$$\eta_i^{(j)}(h) = 0 \text{ when } i \neq j.$$

Let us now form the derivative of $D(x)$, writing this derivative as the sum of n determinants, each of which is obtained from D by differentiating all the elements of one row. By a simple reduction, we find

$$D'(x) = (a_{11} + a_{22} + \ldots + a_{nn})\, D(x).$$

We have, therefore,

$$D(x) = ke^{\int_c^x (a_{11} + \ldots + a_{nn})\, dx},$$

where k is a constant.* Accordingly, $D(c) = k$. But we obviously have $D(c) = 0$. Accordingly, $D(x) \equiv 0$. This, however, contradicts the fact above mentioned that $D(h) \neq 0$. The assumption that there is a point where all the functions η_i do not vanish has thus led us to a contradiction, and thus our theorem is proved.†

The second fundamental theorem which we wish to prove states, roughly speaking, that a small variation in the coefficients of the system (1) and in the initial values c_i produces only a small variation in the solution. The theorem is this:

II. *The positive constants B, C and ε being given, and also a positive function $\phi(x)$ which has at most a finite number of discontinuities in (I) and for which $\int \phi(x)\, dx$ converges when extended over any portion of (I), then a positive δ exists such that if*

$$(1) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{j=n} a_{ij} y_j + \beta_i,$$

$$(\bar{1}) \qquad \frac{d\bar{y}_i}{dx} = \sum_{j=1}^{j=n} \bar{a}_{ij} \bar{y}_j + \bar{\beta}_i, \qquad (i = 1, 2, \ldots . n)$$

are any two systems of equations whose coefficients satisfy condition (A) and also (except possibly at a finite number of points) the inequalities:

$$\begin{array}{lll} |\bar{a}_{ij} - a_{ij}| \leqq \delta, & |a_{ij}| \leqq \phi(x), & |\bar{a}_{ij}| \leqq \phi(x), \\ |\bar{\beta}_i - \beta_i| \leqq \delta, & \left|\int \beta_i\, dx\right| \leqq B, & \left|\int \bar{\beta}_i\, dx\right| \leqq B, \end{array} \qquad (i = 1, 2, \ldots . n)$$

* Cf., for example, Sauvage, "Théorie générale des systèmes d'équations différentielles linéaires et homogènes" (1895), p. 13. This thesis was reprinted from the Annales de la Faculté des Sciences de Toulouse, vols. 8 and 9.

† The proof here given is an immediate generalization of the proof of a similar theorem given by Sturm (Liouville's Journal, vol. I (1836), p. 109), for a single linear differential equation of the second order.

(where the last two ineqalities are supposed to hold no matter what two points of (I) *are taken as the limits of integration), and if* c *is any point of* (I) *and* $(y_1, \ldots . y_n)$ *and* $(\overline{y}_1, \ldots . \overline{y}_n)$ *are solutions of* (1) *and* $(\overline{1})$ *respectively, for which*

$$|\overline{y}_i(c) - y_i(c)| \leqq \delta, \quad |y_i(c)| \leqq C, \quad |\overline{y}_i(c)| \leqq C, \qquad (i = 1, 2, \ldots . n),$$

then

$$|\overline{y}_i(x) - y_i(x)| < \varepsilon, \qquad \begin{pmatrix} i = 1, 2, \ldots . n \\ a \leqq x \leqq b \end{pmatrix}.$$

To prove this theorem, we express the solutions y_i and $\overline{y}_i$ by the method of successive approximations, as explained above, in the form:

$$(3) \qquad y_i = y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \ldots . ,$$
$$(\overline{3}) \qquad \overline{y}_i = \overline{y}_i^{(0)} + \overline{y}_i^{(1)} + \overline{y}_i^{(2)} + \ldots . , \qquad (i = 1, 2, \ldots . n).$$

Writing $B + C = M$, we see that every term of (3) and also of $(\overline{3})$ is in absolute value less than or at most equal to the corresponding term of (12).

Let us choose a positive integer l, so that the remainder of (12), after the l^{th} term, is less than $\varepsilon/3$. Then we have:

$$|\overline{y}_i - y_i| < \sum_{k=0}^{k=l-1} |\overline{y}_i^{(k)} - y_i^{(k)}| + \frac{2\varepsilon}{3}, \qquad (i = 1, 2, \ldots . n).$$

Our theorem will therefore be proved if we can show that δ (which is as yet wholly unrestricted) can be so chosen that

$$|\overline{y}_i^{(k)} - y_i^{(k)}| < \frac{\varepsilon}{3l}, \qquad \begin{pmatrix} i = 1, 2, \ldots . n \\ k = 0, 1, \ldots . -1 \end{pmatrix}.$$

Let us assume that, a positive η being given, a positive δ exists such that

$$|\overline{y}_i^{(k)} - y_i^{(k)}| < \eta, \quad \begin{pmatrix} i = 1, 2, \ldots . n \\ k = 0, 1, \ldots . m < l - 1 \end{pmatrix}. \qquad (15)$$

Then, since we know that such a δ exists, when $k = 0$,* we shall have established our theorem by mathematical induction if we can show that by still further decreasing δ (if necessary) (15) can be made to hold when $k = m + 1$. For this purpose, let us take δ so that

$$\delta < \frac{\eta \,.\, m\,!}{2nM(nN)^m(b-a)}$$

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* We have then merely to take $\delta < \eta/2$, and $\delta < \eta/2\,(b-a)$.

and then let us still further decrease δ (if necessary), so that:

$$|\overline{y}^{(m)} - y_i^{(m)}| < \frac{\eta}{2nN}, \qquad (i = 1, 2, \ldots . n).$$

Now we have:

$$\overline{y}_i^{(m+1)} - y_i^{(m+1)} = \int_c^x \sum_{j=1}^{j=n} (\overline{\alpha}_{ij}\,\overline{y}_j^{(m)} - \alpha_{ij}\,y_j^{(m)})\,dx \qquad (i = 1, 2, \ldots . n).$$

$$= \int_c^x \sum_{j=1}^{j=n} [\overline{\alpha}_{ij}\,(\overline{y}_j^{(m)} - y_j^{(m)}) + y_j^{(m)}\,(\overline{\alpha}_{ij} - \alpha_{ij})]\,dx,$$

Accordingly,

$$|\overline{y}_i^{(m+1)} - y_i^{(m+1)}| \leq \left| \int_c^x \left(n\Phi(x) \cdot \frac{\eta}{2nN} + nM \frac{(nN)^m}{m!} \cdot \frac{\eta \,.\, m!}{2nM\,(nN)^m\,(b-a)} \right) dx \right| \leq \eta,$$

$$(i = 1, 2, \ldots . n).$$

Thus our theorem is proved.

As an immediate application of the theorem just proved, we mention the following theorem:

If, corresponding to the values $k = 1, 2, 3, \ldots .$, we have an infinite number of systems of equations

$$\frac{dy_i^{(k)}}{dx} = \sum_{j=1}^{j=n} \alpha_{ij}^{(k)}\,y_j^{(k)} + \beta_i^{(k)}, \qquad (i, j = 1, 2, \ldots . n), \qquad (16)$$

if the coefficients of each of these systems of equations satisfy conditions (A),* *and if, in a region consisting of all except a finite number of the points of (I), $\alpha_{ij}^{(k)}(x)$ and $\beta_i^{(k)}(x)$ approach uniformly the functions α_{ij} and β_i as k becomes infinite, and if the constants $c_i^{(k)}$ are such that*

$$\lim_{k=\infty} c_i^{(k)} = c_i, \qquad (i = 1, 2, \ldots . n),$$

and if $(y_1, \ldots . y_n)$ and $(y_1^{(k)}, \ldots . y_n^{(k)})$ are solutions of (1) *and* (16) *respectively which satisfy the initial conditions:*

$$y_i(c) = c_i, \quad y_i^{(k)}(c) = c_i^{(k)}, \qquad (i = 1, 2, \ldots . n),$$

then the functions $y_i^{(k)}(x)$ approach uniformly the functions $y_i(x)$ as k becomes infinite.

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* It should be noticed that we need not here exclude the possibility that the number of discontinuities of some or all of the functions $\alpha_{ij}^{(k)}$, $\beta_i^{(k)}$ become infinite with k. Cf. the special case mentioned below.

If, in particular, the coefficients of (1) are continuous throughout (I), we may form the equations (16) by dividing the interval (I) into m_k portions, the integer m_k being supposed to increase without limit as k becomes infinite, and all portions into which (I) is divided being supposed to approach zero as their limits. Let us further denote by $x_1^{(k)}, x_2^{(k)}, \ldots . x_{m_k}^{(k)}$ points lying one in each of these sub-intervals; and let us define $\alpha_{ij}^{(k)}$ and $\beta_i^{(k)}$ throughout the l^{th} of these intervals by the formulæ:

$$\alpha_{ij}^{(k)}(x) = \alpha_{ij}(x_l^{(k)}), \quad \beta_i^{(k)}(x) = \beta_i(x_l^{(k)}).$$

The coefficients of (16) thus consist, for each value of k, of a succession of constant values, so that the functions $y_i^{(k)}$ can be expressed in terms of elementary functions. The method here sketched for approximating to the solution of (1) was developed for a special case at considerable length by Radaković.*

The results of the present paper can be applied without difficulty to the case of a single linear differential equation of the n^{th} order, since such an equation is equivalent, as is well known, to a system of the form (1).†

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

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* Monatshefte für Math. u. Physik, vol. 5 (1894), p. 193.

† The results thus obtained will include, among others, the theorems established by the writer in the Bulletin of the American Mathematical Society for March, 1899, on p. 276.

On the Quaternary Linear Homogeneous Group and the Ternary Linear Fractional Group.

By T. M. Putnam.

§1.—Introduction.

The general group of linear homogeneous substitutions on m indices, with coefficients arbitrary marks of a Galois field such that the determinant is unity, contains an invariant subgroup made up of all those substitutions that multiply every index by an m^{th} root of unity in the field. If $d = [m, p^n - 1]$, then the equation $X^m = 1$ has just d solutions in the field in which we are working,—the $GF[p^n]$, so that this invariant subgroup is of order d.

It has been shown* that the quotient-group defined by this invariant subgroup is simple, and is holoedrically isomorphic with the linear fractional group on $m - 1$ indices. The order of the general homogeneous group of substitutions with determinant unity is

$$M = (p^{nm} - 1)(p^{nm} - p^n)(p^{nm} - p^{2n}) \cdots (p^{nm} - p^{n(m-1)})/(p^n - 1).$$

Hence, the linear fractional group is of order $N = \frac{1}{d} M$.

In a previous paper† the writer has studied the distribution of the substitutions of the homogeneous group into complete sets of conjugate substitutions for the case $m = 4$, and for arbitrary determinant. It is proposed here to consider the same properties when the determinant is unity, both for the homogeneous group and for the fractional group on three indices. The periods of all the substitutions in both groups will be determined and, as far as possible, the arrange-

* Dickson, Annals of Mathematics, vol. XI, pp. 161–183. Burnside, "Theory of Groups," p. 340.

† Amer. Journ. of Math., vol. XXIII, pp. 41–48.

ment of the cyclic subgroups into conjugate sets will be investigated.* Certain commutative subgroups will occur in the investigation which will be studied in their abstract form (cf. §22).

The linear fractional group is further important in that according as p^n is of the form 2^n, $4l-1$, $4l+1$, it is simply isomorphic respectively, with the senary linear groups $FH(6, 2^n)$, $NS(6, p^n)$, $O(6, p^n)$, each defined by a quadratic invariant (Dickson, Bul. A. M. S., 1898–9, p. 472).

§2.—In this case $d = [4, p^n - 1]$, so that

$$d = 1, \text{ if } p^n = 2^n;\ d = 2, \text{ if } p^n = 4l-1;\ d = 4, \text{ if } p^n = 4l+1.$$

Since there are just d solutions of the equation $X^4 = 1$ in the $GF[p^n]$, there are d homogeneous substitutions of determinant unity,

$$\xi_i' = \theta^r (\alpha_{i1}\xi_1 + \alpha_{i2}\xi_2 + \alpha_{i3}\xi_3 + \alpha_{i4}\xi_4), \qquad [i = 1, 2, 3, 4],$$

which, when taken fractionally, lead to the same non-homogeneous substitution of determinant unity, θ^r running through the d roots of the equation $X^4 = 1$.

When $d = 1$, therefore, the linear fractional group G is holoedrically isomorphic with the homogeneous group H. When $d = 4$, we may derive the results for the group G by considering, as identical in H, the substitutions

$$S,\ \Theta S,\ \Theta^2 S,\ \Theta^3 S,$$

where Θ is the substitution multiplying every index by a primitive root θ of the equation $X^4 = 1$ (cf. §11). When $d = 2$, θ and θ^3 are not present in the field, and we may obtain the results for G in this case by regarding S and $\Theta^2 S$ as identical in H.

The discussion falls naturally, then, into the three cases $d = 1$, $d = 2$ and $d = 4$, which will be considered, as far as possible, together, though this will be impracticable in certain cases.

The foundation of the developments here presented rests on the theorem of Jordan,† on the reduction of homogeneous substitutions S on m variables to canonical forms, by transformation of indices, and more directly on its generalization to the $GF[p^n]$ by Professor Dickson.‡ The latter in his paper also sets

* The corresponding properties for the case $n = 3$ has been carried out by Professor Dickson, Amer. Journ. of Math., vol. XXII, pp. 231–252.

† "Traité des Substitutions," pp. 114–126.

‡ Amer. Journ. of Math., vol. XXII, pp. 121-137.

up the corresponding simultaneous canonical form of a substitution commutative with S, of which theorem constant use will also be made.

§3.—If S is the substitution,

$$\begin{aligned} x' &= \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \alpha_{14}w, \\ y' &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + \alpha_{24}w, \\ z' &= \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + \alpha_{34}w, \\ w' &= \alpha_{41}x + \alpha_{42}y + \alpha_{43}z + \alpha_{44}w, \end{aligned}$$

then its characteristic equation $\Delta(\lambda) = 0$ is defined as follows:

$$\Delta(\lambda) \equiv \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{12} & \alpha_{22}-\lambda & \alpha_{23} & \alpha_{24} \\ \alpha_{13} & \alpha_{23} & \alpha_{33}-\lambda & \alpha_{34} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44}-\lambda \end{vmatrix} \equiv \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + 1 = 0,$$

where a, b, c are obviously marks of the $GF[p^n]$.

The substitution

$$x' = ax - \beta y + \gamma z - w, \quad y' = x, \quad z' = y, \quad w' = z$$

has for its characteristic equation

$$\lambda^4 - a\lambda^3 + \beta\lambda^2 - \gamma\lambda + 1 = 0,$$

where α, β, γ are arbitrary marks of the field. Hence they may be so chosen that $\Delta(\lambda)$ is irreducible; the product of a linear factor and an irreducible cubic; the product of two irreducible quadratic factors, distinct or equal; the product of an irreducible quadratic and two linear factors, distinct or equal; or, finally, the product of four linear factors, distinct, or some or all of them equal.

§4.—Following the general theorem cited in §2, we proceed to the enumeration of the canonical forms that will arise. The classification depends on the ways that $\Delta(\lambda)$ breaks up into factors in the field, and the results of §3 show that all possible factorizations will occur, though the equality of the factors will not be possible for certain low values of p^n.

Type 1. When $\Delta(\lambda) = 0$ is irreducible, its roots are of the form λ, λ^{p^n}, $\lambda^{p^{2n}}$, $\lambda^{p^{3n}}$, their product being unity, and λ belonging to the $GF[p^{4n}]$. The corresponding canonical form is

$$x' = \lambda x, \; y' = \lambda^{p^n} y, \; z' = \lambda^{p^{2n}} z, \; w' = \lambda^{p^{3n}} w, \qquad (\lambda^{p^{3n}+p^{2n}+p^n+1} = 1).$$

Type 2. If $\Delta(\lambda)$ has a cubic irreducible factor, the canonical form becomes

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z, \quad w' = \lambda^{-(p^{2n}+p^n+1)} w,$$

where λ, λ^{p^n}, $\lambda^{p^{2n}}$ are the roots of the irreducible cubic factor equated to zero, and x, y, z are conjugate with respect to the $GF[p^n]$.

Type 3. If the equation $\Delta(\lambda) = 0$ breaks up into two distinct irreducible quadratic equations, whose roots are λ, λ^{p^n} and μ, μ^{p^n} respectively, the canonical form becomes

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \mu z, \quad w' = \mu^{p^n} w, \qquad [(\lambda\mu)^{p^n+1} = 1],$$

where $\lambda \neq \mu$ and $\lambda \neq \mu^{p^n}$, x and y, z and w being conjugates.

Type 4. For this case $\Delta(\lambda) = 0$ is the square of an irreducible quadratic with roots λ, λ^{p^n}. Two types of canonical forms will occur, viz.

$$\begin{aligned} &4\,(\text{i}) \quad x' = \lambda x, \quad y' = \lambda(y + x), \quad z' = \lambda^{p^n} z, \quad w' = \lambda^{p^n}(w + z), \\ &4\,(\text{ii}) \quad x' = \lambda x, \quad y' = \lambda y \quad\;\; , \quad z' = \lambda^{p^n} z, \quad w' = \lambda^{p^n} w \quad\;\; , \end{aligned} \qquad [\lambda^{2(p^n+1)} = 1].$$

Type 5. When $\Delta(\lambda)$ is the product of an irreducible quadratic factor and two distinct linear factors, the roots then being λ, λ^{p^n}, α, β, where α and β are in the $GF[p^n]$, the canonical form is

$$x' = \alpha x, \quad y' = \beta y, \quad z' = \lambda z, \quad w' = \lambda^p w, \qquad (\alpha\beta\lambda^{p^n+1} = 1, \alpha \neq \beta).$$

Type 6. If the linear factors in type 5 are equal, i. e. $\alpha = \beta$, two sub-types occur.

$$\begin{aligned} &6\,(\text{i}) \quad x' = \alpha x, \quad y' = \alpha y \quad\;\; , \quad z' = \lambda z, \quad w' = \lambda^{p^n} w, \\ &6\,(\text{ii}) \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \lambda z, \quad w' = \lambda^{p^n} w, \end{aligned} \qquad (\alpha^2\lambda^{p^n+1} = 1).$$

For all the remaining types, the roots of $\Delta(\lambda) = 0$ are in the $GF[p^n]$, the classification being based on the question of their being equal or distinct.

Type 7. If the roots are all distinct the form is

$$x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z, \quad w' = \delta w, \qquad (\alpha\beta\gamma\delta = 1).$$

Type 8. If two roots are equal, there are two forms,

$$\begin{aligned} &8\,(\text{i}) \quad x' = \alpha x, \quad y' = \alpha y \quad\;\; , \quad z' = \beta z, \quad w' = \gamma w, \\ &8\,(\text{ii}) \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \beta z, \quad w' = \gamma w, \end{aligned} \qquad (\alpha^2\beta\gamma = 1).$$

Type 9. When three roots are equal, we have the three forms,

$$\begin{aligned} &9\,(\text{i}) \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \alpha(z + y), \quad w' = \beta w, \\ &9\,(\text{ii}) \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \alpha z \quad\;\; , \quad w' = \beta w, \\ &9\,(\text{iii}) \quad x' = \alpha x, \quad y' = \alpha y \quad\;\; , \quad z' = \alpha z \quad\;\; , \quad w' = \beta w, \end{aligned} \qquad (\alpha^3\beta = 1).$$

Type 10. If all four roots are equal, five cases arise.

$$\begin{aligned}
&10\,(\text{i}) && x' = \alpha x,\ y' = \alpha(y+x),\ z' = \alpha(z+y),\ w' = \alpha(w+z),\\
&10\,(\text{ii}) && x' = \alpha x,\ y' = \alpha(y+x),\ z' = \alpha(z+y),\ w' = \alpha w\ ,\\
&10\,(\text{iii}) && x' = \alpha x,\ y' = \alpha(y+x),\ z' = \alpha z\ ,\ w' = \alpha w\ , && (\alpha^4 = 1),\\
&10\,(\text{iv}) && x' = \alpha x,\ y' = \alpha(y+x),\ z' = \alpha z\ ,\ w' = \alpha(w+z),\\
&10\,(\text{v}) && x' = \alpha x,\ y' = \alpha y\ ,\ z' = \alpha z\ ,\ w' = \alpha w\ ,
\end{aligned}$$

Type 11. If the roots are equal in pairs, there are three cases.

$$\begin{aligned}
&11\,(\text{i}) && x' = \alpha x,\ y' = \alpha y\ ,\ z' = \beta z,\ w' = \beta w\ ,\\
&11\,(\text{ii}) && x' = \alpha x,\ y' = \alpha(y+x),\ z' = \beta z,\ w' = \beta(w+z), && (\alpha^2\beta^2 = 1).\\
&11\,(\text{iii}) && x' = \alpha x,\ y' = \alpha y\ ,\ z' = \beta z,\ w' = \beta(w+z).
\end{aligned}$$

§5.—The above list of canonical forms is complete for the group of substitutions of arbitrary determinant. Furthermore, it is known* that if S and S' are any two substitutions reducible to the same canonical form, there exists a substitution W such that $S' = W^{-1}SW$; i. e. all substitutions reducible to the same canonical form are conjugate within the general group. The converse is also true.

For the group H of substitutions of determinant unity, it is necessary that two substitutions, having the same canonical form, shall be conjugate by means of a substitution of determinant unity. If the determinant of W be ω, and there exists a substitution T of arbitrary determinant τ such that $S = T^{-1}ST$, then choosing $\tau = \omega^{-1}$, the substitution TW will be of determinant unity and is such that

$$(TW)^{-1}S(TW) = S'.$$

It will be shown that the substitution T exists for all types excepting 4 (i), 10 (i), 10 (iv) and 11 (ii). The necessary subtypes will then be set up for these cases.

§6.—The substitution

$$x' = ax,\quad y' = y,\quad z' = z,\quad w' = w$$

is commutative with substitutions of the canonical forms (5), 6 (i), (7), 8 (i),

* Dickson, Amer. Journ. of Math., vol. XXII, p. 136.

11 (i) and 11 (iii), and being of arbitrary determinant, may be taken as the T of the preceding articles for these types.

For type (1), T may be taken as

$$x' = \sigma^r x, \quad y' = \sigma^{rp^n} y, \quad z' = \sigma^{rp^{2n}} z, \quad w' = \sigma^{rp^{3n}} w,$$

where σ is a primitive root of the $GF[p^{4n}]$. Its determinant is $\sigma^{r(p^{3n}+p^{2n}+p^n+1)}$, but since the $(p^{3n}+p^{2n}+p^n+1)^{\text{th}}$ power of σ is a primitive root of the $GF[p^n]$, this determinant is equal to an arbitrary mark of that field.

The substitution

$$x' = x, \quad y' = y, \quad z' = z, \quad w' = aw$$

is commutative with one of types (2), 8 (ii), 9, 10 (ii) and 10 (iii), and is of arbitrary determinant; hence, may be taken as T.

For types (3) and 6 (ii), we may choose as T the substitution

$$x' = x, \quad y' = y, \quad z' = \sigma^r z, \quad w' = \sigma^{rp^n} w,$$

where σ is a primitive root of the $GF[p^{2n}]$. Since σ^{p^n+1} is a primitive root of the $GF[p^n]$, the determinant $\sigma^{r(p^n+1)}$ of this substitution is arbitrary in the $GF[p^n]$.

The most general substitution commutative with a substitution of type 4 (i) has the canonical form

$$x' = \sigma^r x, \quad y' = \sigma^r y + \sigma^s x, \quad z' = \sigma^{rp^n} z, \quad w' = \sigma^{rp^n} w + \sigma^{sp^n} z,$$

where the determinant is $\sigma^{2r(p^n+1)}$, σ being a primitive root of the $GF[p^{2n}]$. Putting $\rho = \sigma^{p^n+1}$, the determinant becomes ρ^{2r}. When $d = 1$, every mark of the $GF[p^n]$ is a square, hence equal to ρ^{2r} for a proper choice of r. When $d = 2$ or 4, only half the marks are squares, so that when ω^{-1} is a not-square, ρ^{2r} cannot be made equal to it. Since the product of a square and a not-square is again a not-square, we may choose r such that $\rho^{2r}\omega^{-1}$ is any particutar not-square t. Hence, any two substitutions S and S' reducible to the canonical form 4 (i), are conjugate by means of a substitution of the field of determinant 1 or t. Hence, for the group of determinant unity two distinct canonical forms arise for this type when $d = 2$ or 4. They may be chosen to be

$$\begin{aligned} &4\,(\text{i}) \quad x' = \lambda x, \quad y' = \lambda(y + x), \quad z' = \lambda^{p^n} z, \quad w' = \lambda^{p^n}(w + z), \\ &4'(\text{i}) \quad x' = \lambda x, \quad y' = \lambda(y + \sigma x), \quad z' = \lambda^{p^n} z, \quad w' = \lambda^{p^n}(w + \sigma^{p^n} z), \end{aligned}$$

where σ is a primitive root of the $GF[p^{2n}]$. These are not conjugate by means of a substitution of determinant unity. For, the most general substitution transforming $4'(1)$ into 4 (i) has the form

$$x' = a\sigma x, \quad y' = a\sigma x + ay, \quad z' = b\sigma^{p^n} z, \quad w' = c\sigma^{p^n} z + bw,$$

where x and z, w and y are conjugates respectively. The determinant of this substitution is seen to be

$$a^2 b^2 \sigma^{p^n+1} = a^2 b^2 \rho;$$

hence, it can never be equal to unity, and, therefore, for the group H, these forms are distinct.

The substitution

$$x' = \sigma^r x, \quad y' = y, \quad z' = \sigma^{rp^n} z, \quad w' = w$$

is commutative with a substitution of type 4 (ii) and its determinant $\sigma^{r(p^n+1)}$, being arbitrary in the $GF[p^n]$, it may be chosen as T for this type.

The most general substitution transforming 10 (i) into itself is

$$x' = ax, \quad y' = bx + ay, \quad z' = cx + by + az, \quad w' = ex + cy + bz + aw,$$

with determinant a^4. If $d = 1$, every mark in the $GF[p^n]$ is a fourth power, so that an a can be found such that $a^4 = \omega^{-1}$. When $d = 2$, $p^n = 4l - 1$. Hence $\rho^{4lk} = \rho^{2k}$, so that squares are also fourth powers. If ω^{-1} is a not-fourth power, then the equation $a^4 = \omega^{-1}.t$ can be satisfied, where t is any particular not-square. Hence, it follows, as for type 4 (i), that when $d = 2$, two substitutions of this canonical form are conjugate by means of a substitution of determinant unity or t. Two subcases then arise which may be taken to be

$$\begin{aligned} &10\,(\mathrm{i}) \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = (z + y), \quad w' = \alpha(w + z), \\ &10'\,(\mathrm{i}) \quad x' = \alpha x, \quad y' = \alpha(y + tx), \quad z' = (z + y), \quad w' = \alpha(w + z). \end{aligned}$$

When $d = 4$, then $p^n = 4l + 1$, so that squares are not always fourth powers as for $d = 2$. If ω is a not-fourth power, then $a^4\omega$ can be made equal to either t, t^2 or t^3, where t is a particular not-square. For, suppose $t = \rho^{4k+1}$, then $t^3 = \rho^{4k'+3}$. Hence, if $\omega = \rho^{4r+1}$, $t\omega^{-1}$ will be a fourth power, hence $a^4 = t\omega^{-1}$ can be satisfied, or $a^4\omega = t$. If $\omega = \rho^{4r+3}$, then $t^3\omega^{-1}$ is a fourth power, while if $\omega = \rho^{2h}$, then $t^2\omega^{-1}$ is a fourth power. Thus when $d = 4$, two substitutions reducible to this canonical form will be conjugate by means of a substitution of deter-

minant 1, t, t^2 or t^3. In addition to types 10 (i) and 10′ (i), there will be two other subcases,

$$10''\text{(i)} \quad x' = \alpha x, \quad y' = \alpha(y + t^2 x), \quad z' = \alpha(z + y), \quad w' = \alpha(w + z),$$
$$10'''\text{(ii)} \quad x' = \alpha x, \quad y' = \alpha(y + t^3 x), \quad z' = \alpha(z + y), \quad w' = \alpha(w + z).$$

These four subtypes are indeed all distinct, for the most general substitution transforming

$$x' = \alpha x, \quad y' = \alpha(y + t^r x), \quad z' = \alpha(z + y), \quad w' = \alpha(w + z)$$

into the substitution

$$x' = \alpha x, \quad y' = \alpha(y + t^s x), \quad z' = \alpha(z + y), \quad w' = \alpha(w + z)$$

is the substitution

$$x' = a_1 x, \quad y' = a_2 x + a_1 t^{s-r} x, \quad z' = a_3 x + a_2 t^{-r} y + a_1 t^{s-r} z,$$
$$w' = a_4 x + a_3 t^{-r} y + a_2 t^{-r} z + a_1 t^{s-r} w,$$

with determinant $a_1^4 t^{3(s-r)}$, which can never be unity for $s,\ r = 0, 1, 2, 3,\ s \neq r$.

The most general substitution transforming type 10 (iv) into itself has a determinant a^2. By a discussion analogous to that used for type 4 (i), we find that when $d = 2$ or 4, two subtypes are necessary which may be taken to be

$$10\text{ (iv)} \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \alpha z, \quad w' = \alpha(w + z),$$
$$10'\text{(iv)} \quad x' = \alpha x, \quad y' = \alpha(y + tx), \quad z' = \alpha z, \quad w' = \alpha(w + z),$$

where t is a particular not-square. These are not conjugate by means of a substitution of determinant unity.

The same state of affairs exists for type 11 (ii), the necessary subtypes being

$$11\text{(ii)} \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \beta z, \quad w' = \beta(w + z),$$
$$11'\text{(ii)} \quad x' = \alpha x, \quad y' = \alpha(y + tx), \quad z' = \beta z, \quad w' = \beta(w + z),$$

where again t is a particular not-square.

§7.—In studying the particular canonical forms, the typical substitution will be denoted by S, and will be thought of as any substitution reducible to the canonical form, or as the canonical form itself. T will designate a substitution commutative with S, while M will denote the order of the homogeneous group H, viz.

$$M = (p^{4n} - 1)(p^{4n} - p^n)(p^{4n} - p^{2n})(p^{4n} - p^{3n})/(p^n - 1),$$

and N will stand for the order of the fractional group G, where $N = \frac{1}{d} M$.

Type 1.

§8.—Among the substitutions of this type occurs

$$S: \quad x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z, \quad w' = \lambda^{p^{3n}} w$$

such that λ is a primitive root of the equation

$$\lambda^{p^{3n}+p^{2n}+p^n+1} = 1.$$

Indeed, by §3, there exists a substitution whose characteristic equation

$$\Delta(\lambda) = \lambda^4 - \alpha\lambda^3 + \beta\lambda^2 - \gamma\lambda + 1 = 0$$

has its coefficients arbitrary marks of the $GF[p^n]$.

Any other substitution of this type will have as multipliers λ^r, λ^{rp^n}, $\lambda^{rp^{2n}}$, $\lambda^{rp^{3n}}$, and is, therefore, a power of S. The period of S is $p^{3n} + p^{2n} + p^n + 1$ while that of the corresponding substitution in G is the least integer m such that

$$\lambda^m = \lambda^{mp^n} = \lambda^{mp^{2n}} = \lambda^{mp^{3n}}.$$

Hence $m(p^n - 1)$ is a multiple of $p^{3n} + p^{2n} + p^n + 1$. But

$$[p^n - 1, p^{3n} + p^{2n} + p^n + 1] = [p^n - 1, 4] = d,$$

so that the period of S in G becomes

$$\frac{1}{d}(p^{3n} + p^{2n} + p^n + 1).$$

Any substitution T commutative with S has the canonical form, simultaneously with the canonical form of S,

$$T: \quad x' = \sigma^r x, \quad y' = \sigma^{rp^n} y, \quad z' = \sigma^{rp^{2n}} z, \quad w' = \sigma^{rp^{3n}} w,$$

where σ is a primitive root of the $GF[p^{4n}]$. But since the determinant is unity, we have

$$\sigma^{r(p^{3n}+p^{2n}+p^n+1)} = 1.$$

Hence r is a multiple of $p^n - 1$. Let $r = k(p^n - 1)$, then, since $\sigma^{p^{4n}-1} = 1$, we may choose $\lambda = \sigma^{p^n-1}$. Hence,

$$\sigma^r = \sigma^{k(p^n-1)} = \lambda^k,$$

so that $T = S^k$. Therefore, S is commutative only with its powers. Denoting by s the number of substitutions commutative with S in H, we have

$$s = p^{3n} + p^{2n} + p^n + 1,$$

while in G the number of commutative substitutions is $\frac{s}{d}$. Consequently, there are $\frac{M}{s}$ substitutions conjugate to S in H, and $\frac{Nd}{s} = \frac{M}{s}$ conjugate to S in G.

The substitutions S, S^{p^n}, $S^{p^{2n}}$, $S^{p^{3n}}$ are the only powers of S that have the same multipliers as S, and are, therefore, the only powers of S that are conjugate to it. Hence, there is one set of $\frac{M}{4s}$ conjugate cyclic subgroups in H of order s, and one set of $\frac{M}{4s}$ in G of order $\frac{s}{d}$.

Remark. The equality of the number of subgroups in the sets of H and of G occurs for most of the types, their orders consequently being in the ratio of d to 1.

To obtain the exact number of substitutions reducible to this type, i. e having an irreducible characteristic equation, we must exclude those powers of S and of its conjugates that have their multipliers in the $GF[p^{2n}]$. This will occur for S^m if $\lambda^{m(p^{2n}-1)} = 1$, hence

$$m(p^{2n} - 1) = h(p^{3n} + p^{2n} + p^n + 1),$$
$$m(p^n - 1) = h(p^{2n} + 1).$$

But $[p^n - 1, p^{2n} + 1] = [p^n - 1, 2]$, which is unity when $d = 1$, and is 2, when $d = 2$ or 4. Therefore,

$$\text{for } d = 1, \quad m = h(p^{2n} + 1), \quad (h = 1, 2, \ldots\ldots p^n + 1),$$
$$\text{for } d = 2, 4, \quad m = \tfrac{1}{2} h(p^{2n} + 1), \quad (h = 1, 2, \ldots\ldots 2(p^n + 1)).$$

We therefore exclude $2j(p^n + 1)$ substitutions from each subgroup of the set in H, and $\frac{2}{d} j(p^n + 1)$ from each subgroup of the set in G, where $j = \frac{1}{2}$ if $d = 1$, and $j = 1$ if $d = 2$ or 4. There remain

$$s - 2j(p^n + 1)$$

substitutions in each subgroup in H; altogether then

$$[s - 2j(p^n + 1)] \frac{M}{4s}.$$

In G there will be left $\frac{1}{d}[s - 2j(p^n+1)]$ substitutions in each cyclic subgroup; in all, therefore,

$$[s - 2j(p^n+1)]\frac{N}{4s}$$

substitutions, each of period a factor of $\frac{s}{d}$, but not of $p^n + 1$, when $d = 1$, and not a factor of $2(p^n + 1)$ when $d = 2$ or 4.

Remark. The number of substitutions reducible to a given canonical form will be determined in every case, and their sum, which must be the order of the group may be calculated as a partial verification of the correctness of the work.

Type 2.

§9. A substitution of this type of determinant unity has the form

$$S: \quad x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z, \quad w' = \lambda^{-(p^{2n}+p^n+1)} w,$$

where λ is in the $GF[p^{3n}]$. Choosing λ to be a primitive root of that field, it follows that any substitution of this type is a power of S. The period of S is $p^{3n}-1$, and the period of the corresponding substitution in G is the least integer m such that

$$\lambda^m = \lambda^{mp^n} = \lambda^{mp^{2n}} = \lambda^{-m(p^{2n}+p^n+1)}.$$

Hence, $m(p^{2n} + p^n + 2) = h(p^{3n} - 1)$, where h is some integer. But

$$[p^{2n} + p^n + 2,\ p^{3n} - 1] = [4,\ p^n - 1] = d,$$

so that $m = \frac{1}{d}(p^n - 1)$.

The most general substitution T of determinant unity, commutative with S, is

$$T: \quad x' = \lambda^r x, \quad y' = \lambda^{rp^n} y, \quad z' = \lambda^{rp^{2n}} z, \quad w' = \lambda^{-r(p^{2n}+p^n+1)}.$$

Hence, $T = S^r$, so that S is commutative only with its powers. It results, then, that there are

$$M/(p^{3n} - 1)$$

substitutions conjugate to S in both H and G. But S, S^{p^n}, $S^{p^{2n}}$ are the only powers of S conjugate to S, so there are $M/3(p^{3n} - 1)$ distinct conjugate cyclic subgroups in the set for both groups.

The substitution S^t will have its characteristic equation further resolvable if

$$\lambda^{t(p^n-1)} = 1;$$

hence, $$t = h(p^{2n} + p^n + 1), \quad h = (1, 2, \ldots . p^n - 1),$$

so that $(p^n - 1)$ powers of S will belong to other canonical forms. There remain $(p^{3n} - p^n)$ substitutions in each subgroup reducible to this canonical form. In all, then, there are in H

$$(p^{3n} - p^n)\, M/3\,(p^{3n} - 1)$$

substitutions, from this type, of period factors of $p^{3n} - 1$, but not factors of $(p^n - 1)$.

The group G evidently contains

$$(p^{3n} - p^n)\, N/3\,(p^{3n} - 1)$$

substitutions from this source, each of period a factor of $\frac{1}{d}(p^{3n} - 1)$, but not of period a factor of $\frac{1}{d}(p^n - 1)$.

Type 3.

§10.—We may choose λ a primitive root of the $GF[p^{2n}]$ in the substitution

$$S: \quad x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \mu z, \quad w' = \mu^{p^n} w \qquad ((\lambda\mu)^{p^n+1} = 1\ \lambda \neq \mu).$$

The period of S is then $p^{2n} - 1$, while the period of S in G is the least integer m such that

$$\lambda^m = \lambda^{mp^n} = \mu^m = \mu^{mp^n}.$$

Hence m is a multiple of $p^n + 1$, say $m = h(p^n + 1)$. But since $(\lambda\mu)^{p^n+1} = 1$, μ is the $[-1 + k(p^n - 1)]^{\text{th}}$ power of λ. Hence

$$\mu^m = \lambda^{-m + km(p^n-1)} = \lambda^m.$$

Hence we have

$$2m - km(p^n - 1) \equiv 0, \qquad (\text{mod } p^{2n} - 1),$$
$$2h(p^n + 1) \equiv 0, \qquad \text{“}$$

Therefore, when $d = 1$, $h = p^n - 1$ and $m = p^{2n} - 1$; and when $d = 2$ or 4, $h = \frac{1}{2}(p^n - 1)$ and $m = \frac{1}{2}(p^{2n} - 1)$.

There are $p^n + 1$ values of μ given by

$$\mu = \lambda^{-1+k(p^n-1)}, \qquad (k = 1, 2, \ldots . p^n + 1).$$

But μ will be in the $GF[p^n]$ if

$$-1 + k(p^n - 1) \equiv 0, \qquad (\text{mod } p^n + 1),$$

which has a solution only when $p = 2$, viz. $k = 2^{n-1}$. Excluding this solution, there remain 2^n values of μ when $d = 1$ and $p^n + 1$, when $d = 2$ or 4. These will give rise to substitutions conjugate in pairs, since, for every value of μ, μ^{p^m} is also in the list given by the values k and $p^n - k$ respectively. Hence, when $d = 1$, there are 2^{n-1} distinct sets of conjugate cyclic subgroups, and when $d = 2$ or 4, there are $\frac{1}{2}(p^n + 1)$ such sets. (The cases $p^n = 2$ and $p^n = 3$ are exceptions, there being no set for the former and only one for the latter.)

The most general substitution T for this type is

$$T: \quad x' = \lambda^r x, \quad y' = \lambda^{rp^m} y, \quad z' = \lambda^s z, \quad w' = \lambda^{sp^m} w,$$

where $\lambda^{(r+s)(p^n+1)} = 1$. Hence, there are $p^n + 1$ values of λ^{r+s}, and for each of these, r may take $p^{2n} - 1$ values and s is then determined, so that there are $(p^n + 1)(p^{2n} - 1)$ substitutions commutative with one of this type. There are then in G

$$M/(p^n + 1)(p^{2n} - 1)$$

substitutions conjugate to S.

Putting $t = -1 + k(p^n - 1)$, the multipliers of S may be written λ, λ^{p^m}, λ^t, λ^{tp^m}. It is seen that S and S^{p^m} are conjugates. Further, they will also be conjugate to S^t and S^{tp^m} if

$$t^2 \equiv 1 \quad \text{or} \quad t^2 \equiv p^n, \qquad (\text{mod } p^{2n} - 1).$$

The first congruence becomes

$$(2k + 1)^2 \equiv 1, \qquad (\text{mod } 2(p^n + 1)),$$

which has $2^{j+\tau}$ distinct solutions, where j is the number of odd prime factors of $2(p^n + 1)$, and if 2^h is the highest power of 2 contained in $2(p^n + 1)$, then $\tau = 0$ if $h = 1$; $\tau = 1$ if $h = 2$; $\tau = 2$ if $h > 2$. (Dirichlet, "Zahlentheorie," §37.)

These $2^{j+\tau}$ solutions for $2k + 1$ may be taken positive and $< 2(p^n + 1)$ and hence give $k \leqq p^n + 1$, so that all must be considered. The corresponding sets, however, will coincide in pairs, for if k is one solution of the congruence $p^n - k$ is also a solution. Hence, just $2^{j+\tau-1}$ sets arise such that powers of the generators of the cyclic subgroups composing them are conjugate by fours.

The other congruence becomes

$$(2k+1)^2 \equiv -1, \qquad (\text{mod } 2(p^n+1)),$$

and has solutions only for $p=2$, $d=1$; it then becomes

$$(2k+1)^2 \equiv -1, \qquad (\text{mod } 2^n+1).$$

But -1 is a residue only of primes of the form $4l+1$. Hence, when 2^n+1 has all its prime factors, j in number, of the form $4l+1$, 2^j solutions occur. These lead to 2^{j-1} sets in our group H.

When $d=1$, therefore, there are g sets of conjugate cyclic subgroups such that any generator is conjugate with four of its powers, where g is $2^{j+\tau-1}+2^{j-1}$, if all the odd prime factors of 2^n+1 are of the form $4l+1$, and g is equal to $2^{j+\tau-1}$ in all other cases. The remaining $2^{n-1}-g$ sets are such that the generators are conjugate with just two of their powers. Hence, in H and in G we have, for $d=1$, g sets each containing

$$M/4(p^n+1)(p^{2n}-1)$$

conjugate cyclic subgroups, and $2^{n-1}-g$ sets each containing

$$M/2(p^n+1)(p^{2n}-1)$$

subgroups; every subgroup being of order $2^{2n}-1$.

When $d=2$ or 4, just $2^{j+\tau-1}$ sets are such that every generator is conjugate with four of its powers, the remaining

$$\tfrac{1}{2}(p^n+1)-2^{j+\tau-1}$$

sets have each generator conjugate with only two of its powers. In H, every group of these sets is of order $p^{2n}-1$.

§11.—In passing from H to G, it must be noticed in each type whether or not S is conjugate with $\Theta^i S$; Θ being the substitution that multiplies every index by θ, a primitive fourth root of unity. In general, if the multipliers of the canonical form of a substitution W are M_1, M_2, M_3, M_4, then those of $\Theta^i W$ are $\theta^i M_1$, $\theta^i M_2$, $\theta^i M_3$, $\theta^i M_4$; and if W is conjugate with $\Theta^i W$, then the two substitutions have the same set of multipliers, apart from their order. Since the determinant of the substitution is unity, we have

$$M_1 M_2 M_3 M_4 = 1.$$

Suppose $\theta^i M_1 = M_2$, then if $\theta^i M_2 = M_1$, $\theta^{2i} = 1$ or $i = 2$; and since $\theta^2 M_3 \neq M_3$, we have $\theta^2 M_3 = M_4$. Hence,

$$M_1 \theta^2 M_1 M_3 \theta^2 M_3 = 1, \text{ or } M_3 = \frac{1}{M_1}, \text{ or } M_3 = \frac{\theta^2}{M_1}.$$

The multipliers are then M_1, $\theta^2 M_1$, $\frac{1}{M_1}$, $\frac{\theta^2}{M_1}$.

If $\theta^i M_2 \neq M_1$, then we must have, by properly choosing the notation, $\theta^i M_2 = M_3$. Hence, the multipliers become M_1, $\theta^i M_1$, $\theta^{2i} M_1$, $\theta^{3i} M_1$ $(i = 1, 2, 3, 4)$. Hence, $M_i^4 = -1$. These are the only two possible sets of multipliers such that W and $\theta^i W$ are conjugates.

In the first case, W and $\Theta^2 W$ would be conjugates and would correspond to the same substitution in G, so that the set of substitutions conjugate to W in H must be divided by two in passing to the group G. If W has the second set of multipliers, then the number of substitutions conjugate to W in H reduces to one-fourth that number for the corresponding set in G.

If $\Theta^i W$ is conjugate with some power of W, for j values of i $(j \leqq 4)$, then the number of subgroups conjugate to the group $\{W\}$ will be reduced in the ratio $\frac{1}{j}$ in passing to the group G. Should it happen that $\Theta^i W$ lies in a set of conjugate cyclic subgroups distinct from the one in which W lies, there will be a corresponding union of sets in G. Since i may be equal to four, both of these latter cases may occur simultaneously, as will happen in type 3 under consideration [cf. §12].

§12.—The multipliers of the typical generator S being λ, λ^{p^n}, λ^t, λ^{tp^n}, where $t = -1 + k(p^n - 1)$, it is seen that they do not fulfill the first conditions of §11. However, it will happen that in passing from H to G some of the sets will coincide when $d = 4$.

There will be the same number of sets in both H and G when $d = 2$, each set containing the same number of conjugate subgroups in the two groups, the orders, however, being $\frac{1}{2}(p^{2n} - 1)$ in the group G. This may be verified by noticing that $\Theta^2 S$ is S raised to the power $\frac{1}{2}(p^{2n} + 1)$.

When $d = 4$, ΘS and $\Theta^3 S$ are not conjugate with powers of S in all cases, and they will be shown in these cases to lie in a set distinct from the one of which S is the typical generator. Hence, in passing from H to G, these two sets unite into one.

To determine the number of such cases, consider when the multipliers of ΘS are the same as those of some power of S. Since $\theta = \lambda^{\frac{1}{4}(p^{2n}-1)}$, the multipliers of ΘS are λ raised to the following powers:

$$\tfrac{1}{4}(p^{2n}+3), \quad p^n + \tfrac{1}{4}(p^{2n}-1), \quad t + \tfrac{1}{4}(p^{2n}-1), \quad tp^n + \tfrac{1}{4}(p^{2n}-1).$$

It will be sufficient to consider S to the powers $\frac{1}{4}(p^{2n}+3)$ and $t + \frac{1}{4}(p^{2n}-1)$, for, if ΘS is conjugate to S^m, it will be conjugate to S^{mp^n}.

The first two multipliers in S to the power $t + \frac{1}{4}(p^{2n}-1)$ coincide with the last two in ΘS. The third will be equal to the first in ΘS if

$$t^2 + \tfrac{1}{4}t(p^{2n}-1) \equiv \tfrac{1}{4}(p^{2n}+3), \qquad [\text{mod } p^{2n}-1],$$

or

$$-2k(k+1) + \tfrac{1}{2}(p^n+1)[-1 + \frac{k}{2}(p^n-1)] \equiv 0, \quad [\text{mod } p^n+1].$$

Since, for $d = 4$, $p^n + 1$ is of the form $2(2l+1)$, this congruence has no solutions, being of the form

$$A + B \equiv 0, \qquad [\text{mod } 2(2l+1)],$$

where A is even and B is odd.

The other possibility is that the third multiplier be equal to the second one of ΘS, which gives the condition

$$t^2 + \frac{t}{4}(p^{2n}-1) \equiv p^n + \tfrac{1}{4}(p^{2n}-1), \qquad [\text{mod } (p^{2n}-1)].$$

Substituting for t its value $-1 + k(p^n-1)$, this becomes

$$-(1 + 2k + 2k^2) + \tfrac{1}{2}(p^n+1)[-1 + \tfrac{1}{2}k(p^n-1)] \equiv 0, \quad [\text{mod } p^n+1].$$

It is, therefore, necessary and sufficient that $1 + 2k + 2k^2$ be an odd multiple of $\frac{1}{2}(p^n+1)$. The congruence

$$2k^2 + 2k + 1 \equiv 0 \qquad [\text{mod } \tfrac{1}{2}(p^n+1)]$$

becomes

$$(2k+1)^2 \equiv -1, \qquad [\text{mod } p^n+1].$$

Since -1 is a residue only of primes of the form $4r+1$, it is necessary that all the odd prime factors of p^n+1 shall be of the form $4r+1$. We will call τ the number of solutions of this congruence that give rise to distinct sets of conjugate subgroups, where τ, if it exists, is an even integer.

Example.—If $p^n = 9$, the above congruence becomes

$$2k^2 + 2k + 1 \equiv 0, \qquad (\text{mod } 5).$$

Then $p^n + 1$, having the single odd prime factor 5, has solutions, viz. $k = 1$ and

$k = 3$. The corresponding substitutions are seen to have the property stipulated.

The other possibility is that the multipliers of ΘS are the same as those of $S^{\frac{1}{4}(p^{2n}+3)}$. The multipliers of the latter are λ raised to the following powers:

$$\tfrac{1}{4}(p^{2n}+3),\quad \tfrac{1}{4}p^n(p^{2n}+3),\quad \tfrac{1}{4}t(p^{2n}+3),\quad \tfrac{1}{4}tp^n(p^{2n}+3).$$

The first two multipliers are the same for the two substitutions, the first ones obviously, the second ones because

$$p^n + \tfrac{1}{4}(p^{2n}-1) \equiv \tfrac{1}{4}p^n(p^{2n}+3), \qquad [\text{mod } (p^{2n}-1)],$$

which reduces to

$$p^n - 1 \equiv 0, \qquad (\text{mod } 4),$$

which is true, since $p^n = 4l+1$. $(d=4)$.

The third multipliers of each will be identical if

$$t + \tfrac{1}{4}(p^{2n}-1) \equiv t \cdot \tfrac{1}{4}(p^{2n}+3), \qquad [\text{mod } p^{2n}-1],$$

Putting $t = -1 + k(p_n - 1)$, this reduces to

$$k(p^n-1) \equiv 2, \qquad [\text{mod } 4],$$

which is not satisfied, since $p^n - 1 \equiv 0 \pmod 4$.

It must then happen that the third multiplier of the first set coincides with the fourth one of the second set, and vice versa. This will be so if

$$t + \frac{p^{2n}-1}{4} \equiv p^n \cdot t\tfrac{1}{4}(p^{2n}+3), \qquad [\text{mod } p^{2n}-1],$$

which reduces to

$$8k + 4 + (p^n+1)^2 - (p^n+1)[kp^n(p^n-1) + 4k] \equiv 0, \quad [\text{mod } 4(p^n+1)],$$

It is, therefore, necessary and sufficient that

$$4(2k+1) + (p^n+1)^2 \equiv 0, \qquad [\text{mod } 4(p^n+1)],$$

which requires that

$$4(2k+1) \equiv 0, \qquad [\text{mod } p^n+1].$$

Since p^n has the form $4l+1$ in this case, $(d=4)$, this may be written

$$2(2k+1) \equiv 0, \qquad [\text{mod } 2l+1],$$

the only solutions of which, $k \leqq p^n + 1$, are

$$k = l \quad \text{and} \quad k = 3l+1 = p^n - l.$$

But, from §10, these two values of k give rise to the same set of conjugate substitutions. Furthermore, these values of k cannot satisfy the congruence [§10]

$$(2k+1)^2 \equiv 1, \qquad [\text{mod } 2(p^n+1)],$$

so that the powers of the generating substitutions in this set are conjugate only by twos.

§13.—When $d=4$, there will be $\tau+1$ sets of conjugate cyclic subgroups such that if S is a typical generator, $\Theta^i S$ $(i=1, 2, 3, 4)$ all lie in the set of which the group $\{S\}$ is a member, τ being zero unless all the prime factors of p^n+1 are of the form $4r+1$. It has been shown that $\Theta^2 S$ is a power of S, hence, ΘS and $\Theta^3 S$ are powers of a certain substitution conjugate to S, so that in the group G the subgroups of these sets, and their orders are just half what they are in the group H. Since there are

$$M/2(p^n+1)(p^{2n}-1)$$

subgroups in each of these sets in H, there will be in G, $\tau+1$ sets having

$$M/4(p^n+1)(p^{2n}-1)$$

conjugate cyclic subgroups, each of order $\frac{1}{2}(p^{2n}-1)$.

There remain in H, $\frac{1}{2}(p^n-1)-\tau$ sets such that the typical substitution S and ΘS lie in different sets, which, therefore, coincide in G [cf. §11], producing

$$\tfrac{1}{4}(p^n-1)-\frac{\tau}{2}$$

sets for that group. From the latter part of §10 it follows that of these sets $2^{l+\tau-3}$ will contain

$$M/4(p^n+1)(p^{2n}-1)$$

cyclic subgroups, while the remainder will have

$$M/2(p^n+1)(p^{2n}-1)$$

subgroups, the order of the subgroups being in all cases $\frac{1}{2}(p^{3n}-1)$.

§14.—*Determination of total number of substitutions reducible to type* 3.

We need only to consider the number of values of λ and μ such that $(\lambda\mu)^{p^n+1}=1$ $(\lambda \neq \mu, \lambda \neq \mu^{p^n})$, and such that each pair of values gives rise to a distinct set of $M/(p^n+1)(p^{2n}-1)$ conjugate substitutions.

(i) $d = 1$. In this case $p = 2$. For every value of λ in the $GF[p^{2n}]$, μ takes $p^n + 1$ values, one of which, by §10, is in the $GF[p^n]$ and is to be excluded. If σ is a primitive root of the $GF[p^{2n}]$ and

$$\lambda = \sigma^{-h(p^n-1)}, \qquad (h = 1, 2, \ldots . p^n).$$

μ may have the values $\sigma^{h(p^n-1)}$ and $\sigma^{hp^n(p^n-1)}$, which must be excluded, since then λ will be equal to μ or μ^{p^n}. Hence, for these p^n values of λ, μ takes only $p^n - 2$ values. For the remaining $p^{2n} - 2p^n$ values of λ not in the $GF[p^n]$, μ may take p^n values so that, in all, there are

$$(p^{2n} - 2p^n)p^n + p^n(p^n - 2) = p^n(p^n + 1)(p^n - 2)$$

values of λ and μ to be considered. The substitutions arising from these sets of values will, however, be conjugate in sets of eight, so that there are

$$\tfrac{1}{8}p^n(p^n + 1)(p^n - 2)$$

distinct sets, each containing

$$M/(p^n + 1)(p^{2n} - 1)$$

distinct conjugate substitutions, each of period a factor of $p^{2n} - 1$ but not of $p^n - 1$.

(ii) $d = 2$ or 4. Let $\lambda = \sigma^t$, then the values of μ are given by

$$\mu = \sigma^{-t + k(p^n-1)}, \qquad (k = 1, 2, \ldots . p^n + 1).$$

But μ will be in the $GF[p^n]$ if

$$-t + k(p^n - 1) \equiv 0, \qquad (\text{mod } p^n + 1),$$

and since $p^n + 1$ is even, this congruence requires that t be even and that

$$2k + t \equiv 0, \qquad (\text{mod } p^n + 1),$$

giving two values of k, $0 < k \leqq p^n + 1$. Hence, for each of the $\frac{1}{2}(p^{2n} - 1)$ values of λ equal to an odd power of σ, μ may take $p^n + 1$ values, making in all $\frac{1}{2}(p^{2n} - 1)(p^n + 1)$.

There are $\frac{1}{2}(p^{2n} - 1) - (p^n - 1) = \frac{1}{2}(p^n - 1)^2$ even powers of σ not in the $GF[p^n]$, and for each of these μ takes $p^n - 1$ values, making in all $\frac{1}{2}(p^n - 1)^3$. The equation

$$\lambda^{2(p^n+1)} = 1$$

has $2(p^n + 1)$ solutions in the $GF[p^{2n}]$, viz.

$$\lambda = \sigma^{k' \frac{p^n-1}{2}}, \qquad (k' = 1, 2, \ldots . 2(p^n + 1)),$$

When $d=2$, the solutions given by $k'=p^n+1$ and $k'=2(p^n+1)$ are in the $GF[p^n]$, and when $d=4$, there are besides these the solutions $k'=\frac{1}{2}(p^n+1)$ and $k'=\frac{3}{2}(p^n+1)$. Hence, there are $2(p^n+1)-d$ solutions not in the $GF[p^n]$. To each one of these solutions correspond two sets of multipliers such that either $\lambda=\mu$ or $\lambda=\mu^{p^n}$, so that we exclude $4(p^n+1)-2d$ of the above solutions, leaving

$$\tfrac{1}{2}(p^{2n}-1)(p^n+1)+\tfrac{1}{2}(p^n-1)^3-4(p^n+1)+2d,$$

or

$$(p^n-1)(p^{2n}+1)-4(p^n+1)+2d$$

sets of solutions λ, μ, which give rise to substitutions conjugate in sets of eight. There are then one-eighth this number of distinct sets of conjugate substitutions of this type for $d=2$ or 4, each set containing

$$M/(p^n+1)(p^{2n}-1)$$

substitutions.

The discussion of §12 shows that to find the number of sets of conjugate substitutions for the group G arising from this type, we need only to divide the number for H, just found, by d, the sets containing the same number of substitutions as those in the group H.

Type 4 (i).

§15.—The general substitution of this canonical form is

$$S:\quad x'=\lambda x,\quad y'=\lambda(y+x),\quad z'=\lambda^{p^n}z,\quad w'=\lambda^{p^n}(w+z),$$

where $\lambda^{2(p^n+1)}=1$. When $d=1$, we may choose λ to be a primitive root of the equivalent equation $\lambda^{p^n+1}=1$. The period of S will then be $p(p^n+1)$ when $d=1$. Denoting by σ a primitive root of the $GF[p^{2n}]$, we may write the most general commutative substitution T in the form

$$x'=\sigma^r x,\quad y'=\sigma^s x+\sigma^t y,\quad z'=\sigma^{rp^n}z,\quad w'=\sigma^{sp^n}z+\sigma^{rp^n}w,$$

where $\sigma^{2r(p^n+1)}=1$, which has for $d=1$ $p^{2n(p^n+1)}$ solutions. Consequently, for $d=1$ there are

$$M/p^{2n}(p^n+1)$$

conjugate with S. But S and $S^{-1}=S^{2p^n+1}$ are conjugate substitutions, so that there is one set of

$$M/2p^{2n}(p^n+1)$$

conjugate cyclic subgroups.

The substitution S^{p^n+1} has all its multipliers in the $GF[p^n]$, while those of the form

$$S^{2h}.(h = 1, 2, \ldots . p^n + 1)$$

are of type 4 (ii), leaving only p^n powers of S and of the generators of the cyclic subgroups conjugate to the group $\{S\}$, that belong to this canonical form. Hence, when $d = 1$, there are in H and in G

$$M/2p^n(p^n + 1)$$

substitutions reducible to this type.

When $d = 2$ or 4, we may choose λ to be a primitive root of the equation

$$\lambda^{2(p^n+1)} = 1,$$

so that S is of period $2p(p^n + 1)$, and any substitution of this canonical form is conjugate with some power of S, just as for $d = 1$. Hence, there is again a single set of conjugate cyclic subgroups.

If we have

$$m = 2k(p^n + 1) + 1, \qquad (k = 1, 2, \ldots . p;\ 2k \not\equiv -1 \pmod{p}),$$

then S and S^m are conjugates. Indeed, the substitution

$$W: \quad x' = \mu x, \quad y' = \sigma x + \mu m^{-1} y, \quad z' = \mu^{p^n} z, \quad w' = \sigma^{p^n} z + \mu^{p^n} m^{-1} w,$$

transforms S^m into S, and since the determinant of W is $\mu^{2(p^n+1)}m^{-2}$, it may be chosen equal to unity, μ being arbitrary in the $GF[p^{2n}]$ and m not being divisible by p.

In the same way it may be shown that S is conjugate with $p - 1$ other powers, the exponents of which are of the form

$$p^n + 2h(p^n + 1), \qquad (h = 1, 2, \ldots . p),$$

where we exclude the one divisible by p. Altogether, then, S is conjugate with $2(p - 1)$ of its powers, and since, for $d = 2$ or 4, there are $2p^{2n}(p^n + 1)$ substitutions T commutative with S, it results that there are

$$M/4(p - 1)p^{2n}(p^n + 1)$$

conjugate subgroups in the set.

In determining the number of substitutions in H reducible to this canonical form for $d = 2$ or 4, we must exclude the $2(p^n + 1)$ powers of the generators of the cyclic groups of the set whose exponents are multiples of p; and also the

$d(p-1)$ powers of S whose exponents are of the form $\frac{2k(p^n+1)}{d}$ that are not contained amongst the above $2(p^n+1)$ excluded powers. There remain

$$2p(p^n+1) - 2(p^n+1) - d(p-1) = (p-1)[2(p^n+1) - d]$$

powers of each generator that belong to this canonical form. Altogether, there are, for $d = 2$ or 4,

$$(2(p^n+1) - d)\, M/4p^{2n}(p^n+1)$$

substitutions in H reducible to this type, their periods being factors of $2p(p^n+1)$ but not factors of $2p$ or $2(p^n+1)$.

The period of S in G is the least integer pr such that

$$\lambda^r = \lambda^{rp^n} \quad \text{or} \quad \lambda^{r(p^n-1)} = 1.$$

Since $[p^n-1, 2(p^n+1)] = d$, it follows that $r = \frac{2}{d}(p^n+1)$. Hence, the period is $\frac{2}{d}p(p^n+1)$, it being so that d powers of each generator in H unite to form a single substitution in G, the number of subgroups in the set remaining unchanged. There are, therefore, in G

$$\frac{1}{d}(p-1)[2(p^n+1) - d]$$

substitutions in each of the subgroups that arise from this canonical form.

Type 4′ (i).

§16.—Since the substitutions of this type are conjugate in the general homogeneous group with those of type 4 (i) by means of a substitution of determinant a particular not-square t, it follows that the two types present the same arrangement as to conjugate cyclic subgroups. Indeed, if $S^{(\nu)}$ represents the substitutions of 4 (i) and W is a properly chosen substitution of determinant t, then $S_1^{(\nu)} = W^{-1}S^{(\nu)}W$ represents the substitutions of 4′ (i) as $S^{(\nu)}$ runs through its list. The substitutions $S_1^{(\nu)}$ being distinct, we see that there are the same number of substitutions for the two types.

Type 4 (ii).

§17.—The general substitution S of this type is

$$x' = \lambda x, \quad y' = \lambda y, \quad z' = \lambda^{p^n} z, \quad w' = \lambda^{p^n} w, \qquad (\lambda^{2(p^n+1)} = 1).$$

Just as in type 4 (i), we choose λ to be a primitive root of the equations

$$\lambda^{(p^n+1)} = 1, \quad \lambda^{2(p^n+1)} = 1,$$

according as $d = 1$ or $d = 2$ or 4 respectively. The substitution S is then of period $p^n + 1$ or $2(p^n + 1)$, as the case may be, and every substitution of this canonical form is a power of S, so that there is a single set of conjugate cyclic subgroups.

The most general substitution T in this case is

$$x' = ax + by, \quad y' = cx + gy, \quad z' = a^{p^n}z + b^{p^n}w, \quad w' = c^{p^n}z + g^{p^n}w,$$

a, b, c, g being marks of the $GF[p^{2n}]$ and $\tau^{p^n+1} = 1$, where $\tau = ag - bc$. Then $\tau^{p^n+1} = 1$ has $p^n + 1$ solutions in the $GF[p^{2n}]$. But for a particular value of τ, the equation $ag - bc = \tau$ has $p^{2n}(p^{4n} - 1)$ solutions. Hence, there are $p^{2n}(p^n + 1)(p^{4n} - 1)$ substitutions commutative with S, and, consequently,

$$M/p^{2n}(p^n + 1)(p^{4n} - 1)$$

conjugate with it.

Since S^{p^n} is the only power of S conjugate to S, there is in the set in H

$$M/2p^{2n}(p^n + 1)(p^{4n} - 1)$$

conjugate cyclic subgroups, and since the period of S in G is $\frac{2}{d}(p^n + 1)$ for $d = 2, 4$, the set in G will also contain this many subgroups.

Excluding the identity, the set contains, when $d = 1$,

$$M/2p^n(p^n + 1)(p^{4n} - 1)$$

substitutions reducible to this type, viz. the p^n powers of S, not the identity, and their conjugates. When $d = 2$ or 4, each group of the set contains $2(p^n+1) - d$ substitutions reducible to this canonical type, viz. those powers of each generator whose multipliers have not one of the d values 1, -1, θ, θ^3.

Type 5.

§18.—Consider the substitution S of this type when λ is a primitive root of the $GF[p^{2n}]$, viz.

$$S: \quad x' = \alpha x, \quad y' = \beta y, \quad z' = \lambda z, \quad w' = \lambda^{p^n}w, \qquad (\alpha\beta\lambda^{p^n+1} = 1, \alpha \neq \beta).$$

If $\alpha = \rho^r$, where $\rho = \lambda^{p^n+1}$ is a primitive root of the $GF[p^n]$, then $\beta = \rho^{-1-r}$. We will have $\alpha = \beta$ if

$$2r + 1 \equiv 0, \qquad (\mathrm{mod}\ p^n - 1),$$

which has a solution only when $p = 2$, i. e. $d = 1$, viz. $r = \frac{1}{2}(p^n - 2)$. Excluding this value which comes under type 6 (ii), r may take $\frac{1}{2}(p^n - 2)$ values, each giving rise to a distinct set of conjugate cyclic subgroups of order $(p^{2n} - 1)$. For $d = 2$ or 4, there are evidently $\frac{1}{2}(p^n - 1)$ such sets.

The most general substitution T is

$$x' = ax, \quad y' = by, \quad z' = \lambda^s z, \quad w' = \lambda^{sp^n} w, \qquad [ab\lambda^{s(p^n+1)} = 1].$$

with $(p^n + 1)(p^n - 1)^2$ distinct forms, so that there are

$$M/(p^n + 1)(p^n - 1)^2$$

substitutions conjugate with one of this type. In the substitution under consideration, S and S^{p^n} are the only powers of S that are conjugates. Hence, in each of the above sets there are

$$M/2(p^n + 1)(p^n - 1)^2$$

cyclic subgroups.

The most general substitution of this canonical form is

$$U: \quad x' = \rho^t x, \quad y' = \rho^{-t-h} y, \quad z' = \lambda^h z, \quad w' = \lambda^{hp^n} w.$$

We will have $U = S^h$ if

$$rh \equiv t, \qquad (\text{mod } p^n - 1),$$

which can be solved for r, provided $[h, p^n - 1]$ is a factor of t. When this condition is not satisfied, sets will arise whose structure may be best studied for particular fields. They will, therefore, be omitted here.

The period of S as a substitution of the group G is the least integer m such that

$$\rho^{rm} = \rho^{-m(r+1)} = \lambda^m = \lambda^{mp^n}.$$

Hence, m is a multiple of $p^n + 1$, say $k(p^n + 1)$. Then, since $\lambda^{p^n+1} = \rho$, we must have, modulo $p^n - 1$,

$$k(p^n + 1)(2r + 1) \equiv 0, \quad rk(p^n + 1) - k \equiv 0.$$

Hence,

$$2k(2r + 1) \equiv 0, \quad k(2r - 1) \equiv 0,$$

which gives the condition $k \equiv 0$ modulo $p^n - 1$. The least value m is then $p^{2n} - 1$, hence the periods of S in H and of S in G being the same, it follows that d substitutions in H unite into one in G, all lying in different subgroups. There will be, then, either a union of d sets into one or a union of d subgroups of the same set into one subgroup, or when $d = 4$, perhaps a combination

of both cases will occur, two sets uniting to form one with half as many subgroups [cf. §11]. Similar properties should be expected also for the cyclic subgroups of this type not included among those of which S is the typical generator.

§19.—*Examples.* As illustrations of what happens for this type, consider some particular fields.

(i) $p^n = 4\,.\,(d = 1)$. There is one substitution S of period 15 with multipliers $1, \rho^2, \lambda, \lambda^4$. The substitution V with multipliers $\rho, \rho^2, \lambda^3, \lambda^5$ is not a power of S, and generates a cyclic group of order 15. There are in this case two sets of conjugate cyclic subgroups of which S and V are the typical generators.

(ii) $p^n = 3\,.\,(d = 2)$. For this case, there is one set of subgroups the typical generator S of period 8, having the multipliers $1, \rho, \lambda, \lambda^3$. Since $\Theta^2 S$ is conjugate with S^5, the set in G contains one-half as many subgroups as the sets in H, the groups in both sets being of order 8.

(iii) $p^n = 5\,.\,(d = 4)$. There are two substitutions S for this case, viz. $(1, \rho^3, \lambda, \lambda^5)$ and $(\rho, \rho^2, \lambda, \lambda^5)$ of period 24. Calling them S_1 and S_2, we see that $\Theta^2 S_1$ is conjugate with S_2^{13} and that ΘS_i and $\Theta^3 S_i$ are conjugate respectively with S_i^7 and S_i^{19}. Hence, the two sets, of which S_1 and S_2 are the typical generating substitutions, coincide in G and the resulting set contains one-half as many subgroups as each of the distinct sets in H contain. Besides S_1 and S_2, H contains the substitution $(\rho, \rho^3, \lambda^4, \lambda^{20})$, which is not contained in the cyclic groups $\{S_1\}$ and $\{S_2\}$. The corresponding set in G is merged into the set resulting from the union of S_1 and S_2.

§20.—In order to determine the arrangement of the substitutions of type 5 into complete sets of conjugate substitutions, and hence find the total number of substitutions reducible to this type, we determine all the solutions of the equation

$$\alpha\beta\lambda^{p^n+1} = 1$$
$$(\alpha \neq \beta,\ \lambda \text{ in the } GF[p^{2n}],\ \text{not in } GF[p^n])$$

that lead to non-conjugate substitutions of the type.

Since λ takes $p^{2n} - p^n$ values not in the $GF[p^n]$ and β may take $p^n - 1$ values not equal to zero, there are $(p^n - 1)(p^{2n} - p^n)$ solutions, including those for $\alpha = \beta$. But the equation

$$\alpha^2\lambda^{p^n+1} = 1$$

gives $p^n + 1$ values for each of the $p^n - 1$ values of α ($\alpha \neq 0$), one of which is in the $GF[p^n]$ when $d = 1$, and two when $d = 2$ or 4. Hence, we must exclude $p^n(p^n - 1)$ substitutions when $d = 1$ and $(p^n - 1)^2$ when $d = 2$ or 4, the excluded ones being exactly those that occur in type 6 (i). There remain, accounting for permutations,

$$\tfrac{1}{4} p^n (p^n - 1)(p^n - 2)$$

solutions for $d = 1$ and

$$\tfrac{1}{4} (p^n - 1)^3$$

for $d = 2$ or 4, which lead to non-conjugate substitutions of this canonical form, each of which is one of a set of $M/(p^n + 1)(p^n - 1)^2$ distinct conjugate substitutions.

Type 6 (i).

§21.—There exists a substitution

$$S: \quad x' = \alpha x, \quad y' = \alpha y, \quad z' = \lambda z, \quad w' = \lambda^{p^n} w, \qquad [\alpha^2 \lambda^{p^n+1} = 1],$$

with λ a primitive root of the $GF[p^{2n}]$. In the case $d = 1$, α is uniquely determined by this choice of λ, and furthermore, every substitution of the type is a power of S. For, consider the most general substitution

$$W: \quad x' = \rho^r x, \quad y' = \rho^r y, \quad z' = \lambda^k z, \quad w' = \lambda^{kp^n} w,$$

where $\rho = \lambda^{p^n+1}$. Since the determinant is unity, we have $\rho^{2r} = \rho^{-k}$. But S^k has the multipliers α^k, α^k, λ^k, λ^{kp^n}, where $\alpha^{2k} = \rho^{-k}$. Hence, $\alpha^{2k} = \rho^{2k}$ which gives $\alpha^k = \rho^r$ when $d = 1$, so that $W = S^k$. There will then be a single set of conjugate cyclic subgroups in this case.

The most general substitution T for this type is

$$x' = ax + by, \quad y' = cx + gy, \quad z' = \lambda z, \quad w' = \lambda^{p^n} w,$$

which can be set up in $p^n(p^{2n} - 1)^2$ distinct ways. Since S and S^{p^n} are the only powers of S conjugate to S, the set contains

$$M/2p^n(p^{2n} - 1)^2$$

conjugate cyclic subgroups of order $p^{2n} - 1$, each group containing $p^{2n} - p^n$ substitutions reducible to this canonical form.

When $d = 2$ or 4, the substitution S as taken above does not exist, since λ^{p^n+1} is a not-square in the $GF[p^n]$. Keeping λ a primitive root of the

$GF[p^{2n}]$, we now choose S to be the substitution,

$$x' = \alpha x, \quad y' = \alpha y, \quad z' = \lambda^2 z, \quad w' = \lambda^{2p^n} w, \qquad (\alpha^2 \lambda^{2(p^n+1)} = 1).$$

We therefore have $\alpha = \pm \rho^{-1}$. Calling A the substitution with $\alpha = +\rho^{-1}$ and B the one with $\alpha = -\rho^{-1}$, we see that for the homogeneous group, both A and B are of period $\frac{1}{2}(p^{2n}-1)$. Furthermore, there exist the relations

$$A^2 = B^2, \quad AB = BA.$$

The cyclic subgroups may be determined by applying the results for the abstract commutative group, which will next be investigated.

§22.—*Determination of the cyclic subgroups of the abstract group defined by the relations*

$$A^m = B^m = 1, \quad AB = BA, \quad A^2 = B^2.$$

No non-cyclic group exists for m an odd integer, for in that case $A^{m+1} = B^{m+1}$, or $A = B$. Assume then that m is even (as it is in type 6 (i)). Suppose, first, that $m = 2m'$, when m' is odd. The cyclic group $\{A\}$ contains all the operators of the form A^t, and $\{B\}$ will have, besides those of the form A^{2t}, all of the form $A^{2t}B$. There remain only those of the type $A^{4t\pm1}B$, which will be present in the group $\{AB\}$. The latter of which is also of order m, when $m = 2m'$, m' odd.

Suppose next that $m = 4m'$, where m' is odd. The cyclic subgroups $\{A\}$ and $\{B\}$ are the same as before; but, since $(AB)^{2m'} = A^{4m'} = 1$, the group $\{AB\}$ is of order $2m'$, and does not contain the operators of the form A^{4t-1}. We must, therefore, have an additional subgroup $\{A^3B\}$ which is of order $\frac{m}{2}$, and contains all the operators of the form $A^{4t-1}B$; for

$$(A^3B)^{2r+1} = A^{8r+3}B,$$

and

$$A^{8r+3} = A^{8r+3-4m'} = A^{8r'-1}, \qquad \left(r' = r - \frac{m'-1}{2}\right).$$

In general, if $m = 2^k m'$, where m' is odd, the smallest number of cyclic subgroups which include all the operators is $k+2$. For example, $\{A\}$, $\{B\}$ of order m, and

$$\{A^{2^j-1}B\} \qquad (j = 1, 2, \ldots . k)$$

of order $\frac{m}{2^j}$ respectively, excepting $\{A^{2^k-1}B\}$, which is of order $\frac{m}{2^{k-1}}$. To prove

this latter statement, suppose that

$$\{A^{2^j-1}B\}^{2t} = A^{2^{j+1}\cdot t} = 1.$$

This will be true if

$$t \cdot 2^{j+1} \equiv 0, \qquad (\text{mod } 2^k m').$$

Hence,

$$t \equiv 0, \qquad (\text{mod } 2^{k-j-1} m').$$

For $j < k$, the least value of $2t$ satisfying this condition is seen to be $\frac{m}{2^j}$, and for $j = k$ it is $\frac{m}{2^{k-1}}$.

§23.—For the subgroups of type 6 we have

$$m = \tfrac{1}{2}(p^{2n} - 1),$$

and since $p^n = 4l \pm 1$, it results that

$$m = 4l(2l \pm 1),$$

so that there are at least four cyclic subgroups.

The multipliers of $A^{2^j-1}B$ being

$$-\rho^{-2^j}, \quad -\rho^{-2^j}, \quad \lambda^{2^{j+1}}, \quad \lambda^{2^{j+1}p^n},$$

it may happen that $\lambda^{2^{j+1}}$ is in the $GF[p^n]$. This can only happen when $p^n + 1$ is a power of 2, and hence only when $d = 2$. Since in that case $p^n + 1$ is always divisible by 2^k, it is necessary that $p^n + 1$ be exactly 2^k. If this condition be satisfied, the values $j = k - 1$ and $j = k$ are such that $\lambda^{2^{j+1}}$ is in the $GF[p^n]$ (e. g. $p^n = 3$, $k = 2$; $p^n = 7$, $k = 3$). We must therefore omit the corresponding cyclic groups, otherwise they would be duplicated in the enumeration of those of type 11 (i).

In general, there will be for H, $k + 2$ sets, each containing $M/2p^n(p^{2n}-1)^2$, conjugate cyclic subgroups, the orders being for A and B, $\frac{1}{2}(p^{2n} - 1)$; for $\{A^{2^j-1}B\}$, $(j = 1, 2, \ldots . k-1)$ it is $2^{-(j+1)}(p^{2n}-1)$; while for $\{A^{2^k-1}B\}$ it is $2^{-k}(p^{2n} - 1)$, where 2^k is highest power of 2 contained in $\frac{1}{2}(p^{2n} - 1)$.

§24.—The period of the substitution A in G is the least integer t such that

$$\lambda^{-t(p^n+1)} = \lambda^{2t} = \lambda^{2lp^n}.$$

Hence t is a multiple of $\frac{1}{2}(p^n + 1)$, say $h\frac{1}{2}(p^n + 1)$. Then

$$\lambda^{-\frac{1}{2}h(p^n+1)^2} = \lambda^{h(p^n+1)},$$

therefore, $$h(p^n+1)[1+\tfrac{1}{2}(p^n+1)] \equiv 0, \qquad (\text{mod } p^{2n}-1),$$

or $$\tfrac{1}{2}h(p^n+3) \equiv 0, \qquad (\text{mod } p^n-1),$$

Since $[p^n+3, p^n-1]=d$, it follows that

$$\tfrac{1}{2}h = \frac{1}{d}(p^n-1), \text{ whence } t = \frac{1}{d}(p^{2n}-1),$$

except when $d=4$ and $k=2$, in which case p^n+3 is divisible by 8, so that the least value of h is $\frac{1}{4}(p^n-1)$ or $t=\frac{1}{8}(p^{2n}-1)$ (cf. §25).

Since t is even in all cases, excepting when it is $\frac{1}{8}(p^{2n}-1)$, it follows that B is of period $\frac{1}{d}(p^{2n}-1)$ in all cases.

The period of the substitution $A^{2^{j-1}}B$ considered in the group G, is the least integer t such that

$$(-\lambda)^{-2^j(p^n+1)\cdot m} = \lambda^{2^{j+1}m} = \lambda^{2^{j+1}p^n \cdot m},$$

which requires that $2^{j+1}m$ be a multiple of p_n+1, say $h(p^n+1)$. We obtain after a slight reduction, the condition

$$h\left\{\frac{p^{2n}-1}{2^{j+2}} - \frac{p^n-1}{2} - 2\right\} \equiv 0, \qquad (\text{mod } p^n-1). \qquad (1)$$

Since $\frac{1}{2}(p_n-1)$ is odd when $d=2$, it follows for $j \lesseqgtr k-2$ that the least integer h satisfying this condition is $h=p^n-1$, the quantity in parenthesis being prime to p^n-1. Hence, the period t of the substitutions

$$A^{2^{j-1}}B \qquad (j \lesseqgtr k-2)$$

is $2^{-(j+1)}\cdot(p^{2n}-1)$.

Where $j=k-1$, the term

$$\frac{p^{2n}-1}{2^{j+2}} = \frac{p^{2n}-1}{2^{k+1}}$$

is odd, so that the bracketed quanity in (1) is even. It results that the least value of h is now $\frac{1}{2}(p^n-1)$, and hence the period for this substitution is $2^{-(k+1)}\cdot(p^{2n}-1)$. Finally, when $j=k$, the congruence (1) may be written

$$\tfrac{1}{2}h\left\{\frac{p^{2n}-1}{2^{k+1}} - (p^n-1) - 4\right\} \equiv 0, \qquad (\text{mod } p^n-1), \qquad (2)$$

showing that $h=2(p^n-1)$ and the period t is then $2^{-k}(p^{2n}-1)$.

The result is that for $d=2$ the periods of all the substitutions remain

unchanged in passing to G, with the one exception $j=k-1$. The corresponding substitution being of period one-half of the period of the substitution in G from which it is derived.

When $d=4$, the congruence (1) may be written

$$h\left\{\frac{p^{2n}-1}{2^{j+3}}-\frac{p^n-1}{4}-1\right\}\equiv 0, \qquad \left(\mathrm{mod}\ \frac{p^n-1}{2}\right), \qquad (3)$$

The highest power of 2 contained in p^n-1 is now 2^k. Hence, when $k>2$, $\frac{p^n-1}{4}$ is even, so that the quantity in the bracket is prime to $\frac{p^n-1}{2}$, for $j\leqq k-2$. Therefore,

$$h=\tfrac{1}{2}(p^n-1), \quad t=2^{-(j+2)}.p^{2n}-1.$$

If $j=k-2$, the quantity in brackets is divisible by 2. Hence,

$$h=\tfrac{1}{4}(p^n-1), \quad t=2^{-(k+1)}.(p^{2n}-1).$$

When $j=k-1$, congruence (3) shows that

$$h=p^n-1, \quad t=2^{-k}(p^{2n}-1),$$

and when $j=k$, writing (3) in the form

$$\tfrac{1}{4}h\left\{\frac{p^{2n}-1}{2^{k+1}}-(p^n-1)-4\right\}\equiv 0, \qquad \mathrm{mod}\frac{p^n-1}{2}\Big).$$

we see that $h=2(p^n-1)$ and $t=2^{-k}(p^{2n}-1)$.

The results, then, for $d=4$ are that the cyclic subgroups in G are of orders one-half the orders of the corresponding groups in H, with the exception of those for which j has the values k, $k-1$, $k-2$, the corresponding groups being of orders, the same for the first two, as in H and for the last, one-fourth of the order of the corresponding group in H.

§25.—The most general substitution of type 6 is of one of the two forms A or A^tB. The multipliers of $A^{t-1}B$ being

$$-\rho^{-t}, \quad -\rho^{-t}, \quad \lambda^{2t}, \quad \lambda^{2tp^n},$$

the multipliers of $\Theta^2A^{t-1}B$ will be the negative of these. Since

$$-\lambda^{2t}=\lambda^{\frac{p^{2n}-1}{2}+2^t},$$

we see that when $d=2$, $\Theta^2 A^{t-1}B = A^{t'}$, where

$$t' = \tfrac{1}{2}\left[\frac{p^{2n}-1}{2} + 2t\right],$$

for $$\rho^{-t'} = \rho^{-(p^n-1)\left(\frac{p^n+1}{4}\right)} . \rho^{-t} = \rho^{-t}.$$

Hence, when $d=2$, any substitution of the type is of the form A^t or $\Theta^2 A^{t'}$. Therefore in G there will be a *single* set of

$$M \mid 2p^n (p^{2n}-1)^2$$

conjugate cyclic subgroups of order $\frac{1}{2}(p^{2n}-1)$.

When $d=4$, the substitution $A^{t-1}B$ is the same as ΘA^r (the case $k=2$ excepted), where $r = t - \frac{1}{8}(p^{2n}-1)$. For, the multipliers of ΘA^r are

$$\theta\rho^{-r}, \quad \theta\rho^{-r}, \quad \lambda^{2t}, \quad \lambda^{2tp^n}$$

Hence, we must have $\theta\rho^{-r} = -\rho^{-t}$. Since $\theta = \rho^{\frac{p^n-1}{4}}$,

$$\theta\rho^{-r} = \theta\rho^{-t}, \quad \rho^{\frac{p^n-1}{4}\cdot\frac{p^n+1}{2}} = \theta \,.\, \rho^{-t} \,.\, \theta^{\frac{p^n+1}{2}} = \theta^2\rho^{-t} = -\rho^{-t}$$

when l is even (i. e $k > 2$), where $p^n = 4l+1$.

When $k=2$, both $\{A\}$ and $\{B\}$ produce distinct cyclic subgroups in G but of order $\frac{1}{8}(p^{2n}-1)$ (cf. §24). In all other cases there is a single set of

$$M \mid 2p^n (p^{2n}-1)^2$$

conjugate cyclic subgroups of order $\frac{1}{4}(p^{2n}-1)$.

In §20 it was shown that the equation

$$\alpha^2 \lambda^{p^n+1} = 1$$

could be satisfied by $p^n(p^n-1)$ values of α and λ when $d=1$, and by $(p^n-1)^2$ when $d=2$ or 4. Accounting for permutations, these solutions produce respectively $\frac{1}{2}p^n(p^n-1)$ and $\frac{1}{2}(p^n-1)^2$ sets of conjugate substitutions, each set containing

$$M \mid p^n (p^{2n}-1)^2$$

distinct substitutions, making a total of

$$\begin{aligned} &M \mid 2(p^n-1)(p^n+1)^2 \quad \text{for} \quad d=1, \\ &M \mid 2p^n (p^n+1)^2 \qquad\quad \text{for} \quad d=2 \end{aligned}$$

distinct substitutions reducible to this type.

Type 6 (ii).

§26 —The general substitution S of this type is

$$x' = ax, \quad y' = a(y + x), \quad z' = \lambda^t z, \quad w' = \lambda^{tp^n} w, \qquad (a^2\lambda^{t(p^n+1)} = 1),$$

where λ is a primitive root of the $GF[p^{2n}]$. When $d = 1$ there will be a single set of conjugate cyclic subgroups of order $2(p^{2n}-1)$, of which the typical generator may be taken as the substitution S with $t = 1$. It is evident that this set will contain all the substitutions of the set of type 6 (i) for $d = 1$.

There are $p^n(p^{2n}-1)$ substitutions commutative with S, viz.

$$x' = ax, \quad y' = bx + ay, \quad z' = \lambda^r z, \quad w' = \lambda^{rp^n} w, \qquad (a^2\lambda^{r(p^n+1)} = 1).$$

The number of substitutions conjugate to S is therefore

$$M/p^n(p^{2n}-1)$$

But S and the $p^{2n} + p^n - 1$ power of S are conjugate since their multipliers are only permuted. Hence there are in the set

$$M|\,2p^n(p^{2n}-1)$$

conjugate cyclic subgroups when $d = 1$.

When $d = 2$ or 4, there will be an arrangement of complete sets of conjugate subgroups similar to that for type 6 (i). The two generators A and B now take the form

$$A: \quad x' = \rho^{-1}x, \quad y' = \rho^{-1}(y + x), \quad z' = \lambda^2 z, \quad w' = \lambda^{2p^n} w,$$
$$B: \quad x' = -\rho^{-1}x, \quad y' = -\rho^{-1}(y + x), \quad z' = \lambda^2 z, \quad w' = \lambda^{2p^n} w,$$

where, as before, $\rho = \lambda^{p^n+1}$. The periods of A and B are both $\frac{1}{2}p(p^{2n}-1)$, and since $AB = BA$ and $A^2 = B^2$, we may apply directly the results of §22, putting

$$m = \tfrac{1}{2}p(p^{2n}-1).$$

The formulation of the number of sets of conjugate cyclic subgroups is the same as for 6 (i), and hence will not be repeated here. It must be noticed, however, that each generator is conjugate with $2(p-1)$ of its powers, viz. all those that have the same multipliers and in which the exponent of the power is not divisible by p (cf. §15).

The number of ways the canonical form can be set up is the same as for

type 6 (i). Hence if $d = 1$, there are $\frac{1}{2} p^n (p^n - 1)$ sets, and when $d = 2$ or 4, there are $\frac{1}{2} (p^n - 1)^2$ sets, each containing

$$M | p^n (p^{2n} - 1)$$

distinct conjugate substitutions.

§27.—When the characteristic equation has all its roots in the $GF[p^n]$, the determinant of the substitutions is $\alpha\beta\gamma\delta = 1$. For the types 7, 8, 9, 10 and 11, it is necessary to know how many solutions there are of this equation, with the assigned equality relations amongst the roots peculiar to the respective canonical forms. The question will first be taken up for $d = 1$.

Solutions of the Equations $\alpha\beta\gamma\delta = 1$ when $d = 1$.

The total number of possible solutions is evidently $(p^n - 1)^3$. If three of the multipliers $\alpha, \beta, \gamma, \delta$ be equal say $\alpha = \beta = \gamma \neq \delta$, then $\alpha^3\delta = 1$, which has $(p^n - 2)$ solutions $\alpha \neq \delta$. Since any three out of the four may be equal, there are in all $4(p^n - 2)$ solutions with just three multipliers equal (cf. type 9). Just two of the multipliers will be equal for six times the number of solutions of

$$\alpha^2\gamma\delta = 1, \qquad (\alpha \neq \gamma \text{ or } \delta).$$

For the $p^n - 2$ values of $\alpha \neq 0$ or 1, γ may take $p^n - 4$ values, since it cannot equal $0, \alpha, \alpha^{-1}$ or α^{-3}. If $\alpha = 1$, γ may take $p^n - 2$ values, $\neq 0$ or 1. For each pair of values of α and γ thus assigned, δ is determined and is unequal to them. Hence there are

$$6[(p^n - 2)(p^n - 4) + (p^n - 2)] = 6(p^n - 2)(p^n - 3)$$

solutions with only two multipliers equal (cf. type 8).

For each of the $p^n - 2$ values of α, $\alpha \neq 1$ or 0 and $\beta = \alpha$, there is one value of γ, viz. $\gamma = \alpha^{-1}$ such that $\delta = \gamma$. Hence, there are $3(p^n - 2)$ sets of solutions with the multipliers equal in pairs (cf. type 11).

Excluding the identity, there remain

$$(p^n - 1)^3 - 4(p^n - 2) - 6(p^n - 2)(p^n - 3) - 3(p^n - 2) - 1 = (p^n - 2)(p^n - 3)(p^n - 4)$$

sets with all the multipliers distinct (cf. type 7).

§28.—*Solutions of the Equation $\alpha\beta\gamma\delta = 1$ when $d = 2$ or 4.*

For the $p^n - d - 1$ values of α, $\alpha \neq 0$, and not equal to d of the quantities $1, \theta, \theta^2, \theta^3$ present in the field, the mark γ takes $p^n - 5$ values, satisfying the

condition that $\gamma \neq \alpha$, but $\gamma = \delta$, the excluded values of γ being $0, \pm \alpha^{-1}, \alpha, \alpha^{-3}$. For the d values of α, viz. $1, \theta, \theta^2, \theta^3$, γ takes $p^n - 3$ values, for in this case $\theta^3 = \theta$, $\theta^{-1} = \theta^3$ and $-\theta^{-1} = \theta$, so that only three values for γ must be excluded. Allowing for permutations, there are then

$$6(p^n - d - 1)(p^n - 5) + 6d(p^n - 3)$$

sets of solutions with only two multipliers equal.

For every value of $\alpha \neq 1, \theta, \theta^2, \theta^3$, there are two values of γ, viz. $\pm \alpha^{-1}$, for which $\alpha = \beta$ and $\gamma = \delta$, $(\alpha \neq \gamma)$; and for each of the d values $1, \theta, \theta^2, \theta^3$, there is one such solution. Hence there will be

$$6(p^n - d - 1) + 3d$$

sets of multipliers equal in pairs. There will be $4(p^n - 1 - d)$ with just three equal and d sets with all four equal. There remain, therefore,

$$(p^n - 1)^3 - 4(p^n - d - 1) - 6(p^n - d - 1)(p^n - 5) - 6d(p^n - 3) - 6(p^n - d - 1) - 4d,$$

or

$$p^{3n} - 9p^{2n} + 29p^n - 6d - 21$$

sets with all of the multipliers distinct. (This formula does not hold for $d = 1$, as may be seen by comparing the two results.)

Type 7

§29.—Since four unequal multipliers may be permuted in 24 distinct ways, it follows from the preceding articles that there are $\frac{1}{24}g$ distinct substitutions of this type where

$$g = (p^n - 2)(p^n - 3)(p^n - 4), \qquad (d = 1),$$
$$g = p^{3n} - 9p^{2n} + 29p^n - 6d - 21, \qquad (d = 2 \text{ or } 4).$$

There are $(p^n - 1)^3$ commutative substitutions for this type, viz.

$$x' = ax, \quad y' = by, \quad z' = cz, \quad w' = ew, \qquad (abce = 1).$$

Hence, there are $\frac{1}{24}g$ sets of conjugate substitutions, each set containing $M/(p^n - 1)^3$ substitutions of period a factor of $p^n - 1$.

The number of distinct substitutions of this type of period exactly $p^n - 1$, can be shown to be

$$\tfrac{1}{24}[F_3(p^n - 1) - 6F_2(p^n - 1) + 11F_1(p^n - 1)],$$

where

$$F_i(p^n-1) = (p^n-1)^i\left(1-\frac{1}{p_1^i}\right)\left(1-\frac{1}{p_2^i}\right)\cdots\left(1-\frac{1}{p_k^i}\right),$$

$p_1 \cdots p_k$ being the distinct prime factors of p^n-1. $F_i(p^n-1)$ designates the number of ways of choosing i integers equal to or less than p^n-1, the different permutations of the i integers being considered, and such that their greatest common divisor with p^n-1 is unity.*

Type 8 (i).

§30. S: $x' = \alpha x, \quad y' = \alpha y, \quad z' = \beta z, \quad w' = \gamma w, \qquad (\alpha^2\beta\gamma = 1).$

In §§27, 28, the number of sets of multipliers, say q, for this type was shown to be

$$\begin{aligned} &6(p^n-2)(p^n-3), && (d=1),\\ &6\left[(p^n-1)(p^n-5)+2d\right], && (d=2 \text{ or } 4). \end{aligned}$$

But four letters, two of which are equal, may be permuted in 12 distinct ways.

Hence there are $\frac{q}{12}$ sets of conjugate substitutions.

There are $p^n(p^n-1)^3(p^n+1)$ substitutions commutative with S, viz.

$$x' = ax+by, \quad y' = a'x+b'y, \quad z' = cz, \quad w' = ew,$$
$$[ce(ab'-a'b)] = 1.$$

Hence each one of the $\frac{q}{12}$ sets contains

$$M/p^n(p^n-1)^3(p^n+1)$$

conjugate substitutions of period a factor of p^n-1.

Type 8 (ii).

§31.—There are $p^n(p^n-1)^2$ substitutions commutative with a substitution of this type, viz. all of those of the form

$$x' = ax, \quad y' = bx+ay, \quad z' = cz, \quad w' = ew, \qquad (abc^2 = 1).$$

Hence the $\frac{q}{12}$ sets for this type each contain

$$M/p^n(p^n-1)^2$$

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* Jordan, "Traité des Substitutions," p. 96.

conjugate substitutions each of period a factor of $p(p^n-1)$.

Type 9 (i).

§32.—A substitution

$$S: \quad x'=\rho x, \quad y'=\rho(y+x), \quad z'=\rho(z+y), \quad w'=\rho^{-3}w$$

exists for which ρ is a primitive root of the $GF[p^n]$. Since S^m is

$$x'=\rho^m x, \quad y'=\rho^m(y+mx), \quad z'=\rho^m\left(z+my+\tfrac{1}{2}m(m-1)z\right), \quad w'=\rho^{-3m}w$$

the period of S is $p(p^n-1)$ for $p>2$, and for $p=2$ it is $4(2^n-1)$.

The most general substitution W of this type has the form

$$x'=\rho^k x, \quad y'=\rho^k(y+x), \quad z'=\rho^k(z+y), \quad w'=\rho^{-3k}w$$

Hence W is the power $(k-1)(p^n-1)+k$ of S. For

$$\rho^{(k-1)(p^n-1)+k}=\rho^k$$

and
$$(k-1)(p^n-1)+k\equiv 1, \pmod{p}.$$

Therefore there is a single set of conjugate cyclic subgroups for this type.

There are $p^{2n}(p^n-1)$ substitutions commutative with S, viz.

$$x'=ax, \quad y'=bx+ay, \quad z'=cx+by+az, \quad w'=ew, \qquad (a^3e=1),$$

and hence
$$M/p^{2n}(p^n-1)$$
conjugate with it.

When $d=1$, S and S^{2p^n-1} are conjugates, and when $d=2$ or 4, S will be conjugate with all of its powers such that

$$m\equiv 1 \pmod{p^n-1}, \quad m\not\equiv 0 \pmod{p}.$$

Hence m has the form $h(p^n-1)+1$, where h can take all the values $2, 3 \ldots p$; so S is conjugate with $(p-1)$ of its powers. Indeed, a substitution V transforming S into S^m (m not divisible by p) is

$$x'=x, \quad y'=my, \quad z'=x+\tfrac{1}{2}m(m-1)y+m^2z, \quad w'=aw, \qquad (m^3a=1).$$

Hence, when $d=1$, the set contains

$$M/2p^{2n}(p^n-1)$$

conjugate cyclic subgroups of order $4(2^n-1)$, and when $d=2$ or 4, it has

$$M/(p-1)p^{3n}(p^n-1)$$

such subgroups, each being of order $p(p^n-1)$.

The period of S in G for $d > 1$ is the least integer t. p such that

$$\rho^t = \rho^{-3t} \quad \text{or} \quad \rho^{4t} = 1.$$

Hence $t = \frac{1}{d}(p^n - 1)$ and $pt = \frac{1}{d} p(p^n - 1)$. Therefore the set in G contains the same number of cyclic subgroups as the set in H, but of orders reduced in the ratio $1 : d$.

From the results of §§26, 27, it follows that there are in H, $p^n - d - 1$ distinct sets each containing

$$M/p^{2n}(p^n - 1)$$

conjugate substitutions, and in G there are $\frac{1}{d}(p^n - d - 1)$ such sets.

Type 9 (ii).

§33.—There will be a single set of conjugate substitutions for this type, of which the typical generator is

$$S: \quad x' = \rho x, \quad y' = \rho(y + x), \quad z' = \rho z, \quad w' = \rho^{-3} w,$$

where again ρ is a primitive root of the $GF[p^n]$. The period of S is $p(p^n - 1)$ for all cases, and the period of the corresponding substitution in G is $\frac{1}{d} p(p^n - 1)$.

There are $p^{3n}(p^n - 1)^2$ substitutions commutative with S, viz.

$$x' = ax, \quad y' = bx + ay + cz, \quad z' = ex + fz, \quad w' = gw, \qquad (a^2 fg = 1),$$

and hence there are

$$M/p^{3n}(p^n - 1)^2$$

substitutions conjugate to S. But S is conjugate with $p - 1$ of its powers, viz. those whose exponents are

$$m = h(p^n - 1) + 1, \qquad (h = 2, 3, \ldots . p).$$

Indeed, the substitution

$$x' = x, \quad y' = my, \quad z' = z, \quad w' = aw \qquad (am = 1)$$

transforms S into S^m. Hence there are

$$M/(p - 1)p^{3n}(p^n - 1)^2$$

conjugate cyclic subgroups in the set.

From §§26, 27, it follows that there are in H $p^n - d - 1$ sets of conjugate substitutions for this type, each set containing

$$M/p^{3n}(p^n - 1)^2$$

substitutions, and in G there are $\frac{1}{d}(p^n - d - 1)$ such sets.

Type 9 (iii).

§34.—The typical generator in this case is

$$S: \quad x' = \rho x, \quad y' = \rho y, \quad z' = \rho z, \quad w' = \rho^{-3} w$$

of period $p^n - 1$ in H and $\frac{1}{d}(p^n - 1)$ in G.

The commutative substitution T in this case will have the form

$$x' = a_1 x + b_1 y + c_1 z, \quad y' = a_2 x + b_2 y + c_2 z, \quad z' = a_3 x + b_3 y + c_3 z, \quad w' = ew$$

with determinant

$$e \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1.$$

The number of ways this determinant can be set up is equal to the order of the general ternary group, viz.

$$s = (p^{3n} - 1)(p^{3n} - p^n)(p^{3n} - p^{2n}).$$

Since s is not conjugate with any of its powers, there are M/s conjugate cyclic subgroups in the set, each group containing $p^n - d - 1$ substitutions reducible to this canonical form. In G there will be the same arrangement of cyclic subgroups with the order of each $\frac{1}{d}(p^n - 1)$ instead of $p^n - 1$.

Type 10 (i).

§35.—The general substitution S of this type has the form

$$x' = \alpha x, \quad y = \alpha(y + x), \quad z' = \alpha(z + y), \quad w' = \alpha(w + z), \qquad (\alpha^4 = 1).$$

When $d = 1$, $\alpha = 1$; $d = 2$, $\alpha = \pm 1$, and when $d = 4$, $\alpha = 1, \theta, \theta^2$ or θ^3, where θ denotes always a primitive root of the equation $d^4 = 1$. Hence, there

are in H, d sets of conjugate substitutions. To find the period of S, consider S^m, viz.

$$x' = \alpha^m x, \quad y' = \alpha^m (y + mx), \quad z' = \alpha^m \left(z + my + \frac{m(m-1)}{2} x \right),$$

$$w' = \alpha^m \left(w + mz + \frac{m(m-1)}{2} y + \frac{m(m-1)(m-2)}{6} x \right).$$

When $d = 1$, i. e. $p = 2$ and $\alpha = 1$, the period is evidently 4; for $p = 3$ the period is 9 for $\alpha = +1$ and 18 for $\alpha = -1$. For $d = 2$, $p \neq 3$, the period is p or $2p$, according as $\alpha = +1$ or $\alpha = -1$. When $d = 4$, the period is p, $4p$, $2p$ or $4p$, according as $\alpha = 1, \theta, \theta^2$ or θ^3 respectively.

The following substitution W

$$x' = a_1 x, \quad y' = a_2 x + m a_1 y, \quad z' = a_3 x + \tfrac{1}{2}[2m a_2 + m(m-1) a_1] y + m^2 a_1 z,$$
$$w' = a_4 x + [m a_3 + \tfrac{1}{2} m(m-1) a_2 + \tfrac{1}{6} m(m-1)(m-2) a_1] y$$
$$+ \tfrac{1}{2}[2m^2 a_2 + m^2(m-1) a_1 + m^2(m-1) a_1] z + m^3 a_1 w$$

is the most general substitution transforming S into S^m. The determinant of W being $m^6 a_1^4$, is different from zero only when m is not a multiple of p, and in that case the determinant can be made equal to unity by properly choosing a_1. Hence S is conjugate with all its powers, the exponents of which are not multiples of p.

There are dp^{3n} substitutions commutative with S, viz.

$$x' = ax, \quad y' = bx + ay, \quad z' = cx + by + az, \quad w' = fx + cy + bz + az, \quad (a^4 = 1).$$

Hence, there are M/dp^{3n} substitutions conjugate to S.

In the particular case $p = 3$, the substitutions S, S^2, S^4, S^5, S^7 and S^8 are conjugates when $\alpha = +1$, making in the set $\frac{M}{6dp^{3n}} = \frac{M}{324}$ conjugate cyclic subgroups of order 9. When $p = 3$ and $\alpha = -1$, S is conjugate with 12 of its powers, giving them a set of $\frac{M}{648}$ subgroups of order 18.

When $d = 2$, $p > 3$, the substitution S is conjugate with $p-1$ of its powers when $\alpha = 1$, and with $2(p-1)$ when $\alpha = -1$. It results, therefore, that the corresponding sets contain respectively

$$M/2(p-1)p^{3n}, \quad M/4(p-1)p^{3n}$$

conjugate cyclic subgroups with orders p and $2p$.

When $d=4$, S is conjugate with $p-1$, $2(p-1)$ or $4(p-1)$ of its powers according as α is $+1$, -1 or θ or θ^3. The four sets have, then, respectively,

$$M/4(p-1)p^{3n}, \quad M/8(p-1)p^{3n}, \quad M/16(p-1)p^{3n}$$

conjugate cyclic subgroups.

Since in each of the d sets of conjugate substitutions, there are M/dp^{3n} substitutions, the total number of substitutions reducible to this canonical form is M/p^{3n}.

§36.—The period of S in the fractional group G is independent of α, and is equal to 4 when $d=1$; for $d=2$ and $p=3$, it is 9; for $d=2$, $p\neq 3$, and for $d=4$, it is p. In all cases there is a single set of conjugate cyclic subgroups. The number of substitutions conjugate to S in G is M/p^{3n}, the same as in H. For $d=1$, the structure of the set is, of course, the same in both groups, and generally the set in G is abstractly identical with the corresponding set in H arising for $\alpha=1$. In passing to G, the d sets of H are absorbed into this one.

By a discussion analogous to that of §16, the d subtypes 10(i), 10′(i), 10″(i) and 10‴(i) that are present may each be shown to have a structure identical with that of 10(i) just discussed. Hence, the group G has d sets of conjugate cyclic subgroups arising from this type.

Type 10(ii).

§37.—There are d sets of conjugate cyclic subgroups for this type of which the general substitution S is

$$x'=\alpha x, \quad y'=\alpha(y+x), \quad z'=\alpha(z+y), \quad w'=\alpha w, \qquad (\alpha^4=1).$$

The period of S is 4 when $p=2$ (i. e. $d=1$ and $\alpha=1$); for $\alpha=1$, $p\neq 2$, the period is p; for $\alpha=-1$, it is $2p$, and for $\alpha=\theta$ or θ^3, it is $4p$ (cf. §35).

If m is prime to p, the substitution

$$x'=x, \quad y'=m^{-1}y, \quad z'=z-\frac{m-1}{2m}y, \quad w'=mw$$

transforms S into S^m. Hence, for $p=2$, S and S^3 are conjugates. If $\alpha=1$, $p\neq 2$, S is conjugate with $p-1$ of its powers, if $\alpha=-1$, with $2(p-1)$ powers, and if $\alpha=\theta$ or θ^3, with $4(p-1)$ of its powers. Since there are

$p^{4n}(p^n-1)$ substitutions

$$x'=a_1x,\ y'=a_2x+a_1y,\ z'=a_3x+a_2y+a_1z+a_4w,\ w'=a_5x+a_6w,\ (a^3f=1),$$

it follows that the set for $p=2$, $(d=1)$ contains

$$M/2p^{4n}(p^n-1)$$

subgroups; when $\alpha=1$, $p\neq 2$, the set contains

$$M/(p-1)p^{4n}(p^n-1)$$

subgroups; when $\alpha=-1$, there are

$$M/2(p-1)p^{4n}(p^n-1);$$

and when $\alpha=\theta$ or θ^3, there will be

$$M/4(p-1)p^{4n}(p^n-1)$$

subgroups in the set.

Since there are $M/p^{4n}(p^n-1)$ substitutions in each of the d complete sets of conjugate substitutions, it follows that there are in H

$$dM/p^{4n}(p^n-1)$$

substitutions reducible to this canonical form.

The structure of the set in G is again the same as the set in H given by $\alpha=1$ (cf. §36), the d sets of the group H being merged into a single set in G.

Type 10 (iii)

§38.—As in the two preceding cases, there are d types corresponding to the d solutions of $\alpha^4=1$, viz.

$$S:\quad x'=\alpha x,\quad y'=\alpha(y+x),\quad z'=\alpha z,\quad w'=\alpha w,\qquad (\alpha^4=1).$$

If $\alpha=1$, the period is equal to p for all cases. The periods for the other values of α are the same as in type 10 (ii), and here also S will be conjugate with all its powers, the exponents of which are prime to p.

There are $\qquad s\equiv p^{5n}(p^{2n}-1)(p^{2n}-p^n)$

substitutions commutative with S, viz.

$$x'=a_1x,\quad y'=a_2x+a_1y+a_3z+a_4w,$$
$$z'=a_5x+a_6z+a_7w,\quad w=a_8x+a_9y+a_{10}w,\qquad (a_1^2\Delta=1),$$

where $\Delta = (a_6 a_{10} - a_7 a_9)$. For each of the $p^n - 1$ values of $a_1 \not\equiv 0$, Δ is determined, so that there are $(p^{2n}-1)(p^{2n}-p^n)$ corresponding sets of values for a_6, a_{10}, a_7, a_9. Hence there are s solutions of $a_1^2\Delta = 1$.

When $\alpha = 1$, therefore, there are

$$M/s\,(p-1)$$

conjugate cyclic subgroups in the set; when $\alpha = -1$, or $\alpha = \theta$ or θ^3, there are respectively

$$M/2s\,(p-1) \quad \text{and} \quad M/4s\,(p-1)$$

such subgroups. The results for G will again be gotten by putting $\alpha = 1$, giving in all cases one set of conjugate cyclic subgroups.

Type 10 (iv).

§39.—There are d sets for this type, of which the general substitution is

$$S: \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \alpha z, \quad w' = \alpha(w + z),$$

the period of S being in all cases exactly as in the preceding type, and S being conjugate with all its powers, the exponents of which are not divisible by p.

The most general substitution T commutative with S has the canonical form

$$x' = a_1 x + a_2 z, \quad y' = a_3 x + a_1 y + a_4 z + a_2 w,$$
$$z' = a_6 x + a_7 z. \quad w' = a_8 x + a_6 y + a_{10} z + a_7 w$$

with determinant

$$\Delta \equiv \begin{vmatrix} a_1 & 0 & a_2 & 0 \\ a_3 & a_1 & a_4 & a_2 \\ a_6 & 0 & a_7 & 0 \\ a_8 & a_6 & a_{10} & a_7 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & 0 & 0 \\ a_6 & a_7 & 0 & 0 \\ a_3 & a_4 & a_1 & a_2 \\ a_8 & a_{10} & a_6 & a_7 \end{vmatrix} \equiv (a_1 a_7 - a_2 a_6)^2 = 1$$

Hence when $d = 1$, there are

$$s = p^{5n}(p^{2n} - 1)$$

substitutions commutative with S, and when $d = 2$ or 4 there are 2 such substitutions, and consequently M/s and $M/2s$ substitutions conjugate with S respectively. Therefore, when $d = 1$ the set contains M/s subgroups, when $\alpha = 1$, $d \not\equiv 1$, it contains $M/2s\,(p-1)$ subgroups; when $\alpha = -1$, it has $M/4\,(p-1)\,s$, and when $\alpha = \theta$ or θ^3, there will be $M/8\,(p-1)\,s$ conjugate cyclic subgroups.

Since the structure of type 10′ (iv) is exactly the same as for this type, eacn of the above results are duplicated for that type when $d = 2$ or 4.

The group G contains a single set of conjugate cyclic subgroups from type 10 (iv) abstractly identical with the set in H given by $\alpha = 1$. When $d = 2$ or 4, 10′ (iv) also produces a set in G distinct from the preceding one.

Altogether, types 10 (iv) and 10′ (iv) give rise to dM/s substitutions in H and to M/s substitutions in G.

Type 10 (v).

§40.—The substitution

$$x' = \alpha x, \quad y' = \alpha y, \quad z' = \alpha z, \quad w' = \alpha w, \qquad (\alpha^4 = 1)$$

is one of the substitutions Θ^i, α taking d of the values $1, \theta, \theta^2, \theta^3$. These d sub stitutions form precisely the invariant subgroup that defines the isomorphism between H and G.

Type 11 (i).

§41.—The general substitution of this type has the canonical form

$$S: \quad x' = \alpha x, \quad y' = \alpha y, \quad z' = \beta z, \quad w' = \beta w, \qquad (\alpha^2\beta^2 = 1, \ \alpha \neq \beta)$$

the period of which is $p^n - 1$. If ρ is a primitive root of the $GF[p^n]$, then when $d = 1$, every substitution of the above form is a power of the substitution with multipliers $\rho, \rho, \rho^{-1}, \rho^{-1}$. There will be in H and in G, therefore, a single set of conjugate cyclic subgroups.

The most general substitution commutative with one of this type is

$$x' = a_1x + b_1y. \quad y' = a_2x + b_2y, \quad z' = c_1z + d_1w, \quad w' = c_2z + d_2w,$$

with determinant

$$(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1) = 1.$$

One of the factors may be chosen arbitrarily in $(p^{2n} - 1)(p^{2n} - p^n)$ ways; the other factor, being then determined by the above relation, can be satisfied in $p^n(p^{2n} - 1)$ ways. Hence in all there are

$$s = (p^{2n} - p^n)^2(p^n + 1)^2(p^n - 1)$$

commutative substitutions. Since the only power of S conjugate to S is S^{-1}, there will be in H and in G for $d = 1$, $M/2s$ conjugate cyclic subgroups in the set, each subgroup containing $p^n - 2$ substitutions reducible to this type, (cf. §27).

Type 11 (i) *for* $d = 2$ *or* 4.

§42.—In this case there will not be a single set of conjugate cyclic subgroups in H. Let A be the substitution

$$A: \quad x' = \rho x, \quad y' = \rho y, \quad z' = \rho^{-1} z, \quad w' = \rho^{-1} w,$$

and let B be the substitution

$$B: \quad x' = \rho x, \quad y' = \rho y, \quad z' = -\rho^{-1} z, \quad w' = -\rho^{-1} w.$$

Any substitution in this canonical form is then representable either in the form A^t or $A^t B$. This can be verified by noticing that the multipliers of the general substitution are

$$\rho^h, \rho^h \pm \rho^{-h}, \pm \rho^{-h},$$

which, with the plus sign, are the multipliers of A^h and with the minus sign are those of $A^{h-1}B$. Furthermore, A and B are of period $p^n - 1$ and fulfil the relations

$$AB = BA, \quad A^2 = B^2,$$

We may therefore apply directly the results of §22 to determine the cyclic subgroups.

Putting then the m of §20 equal to $p^n - 1$, k becomes the highest power of 2 contained in $p^n - 1$, so that $k = 1$ if $d = 2$, and there are in that case three cyclic subgroups

$$\{A\}, \quad \{B\}, \quad \{AB\},$$

each of order $p^n - 1$. When $d = 4$, $k \geqq 2$, and there are $k + 2$ cyclic subgroups, $\{A\}$ and $\{B\}$ of order $(p^n - 1)$ and

$$\{A^{2^j - 1} B\}, \qquad (j = 1, 2, \ldots . k - 1)$$

of order $2^{-j}(p^n - 1)$ respectively, and, finally, $\{A^{2^k - 1} B\}$ of order $2^{-(k+1)}(p^n - 1)$.

The substitution A is conjugate with its reciprocal. Consider the substitution $A^{2^j - 1} B$. Its multipliers are

$$\rho^{2^j}, \quad \rho^{2^j}, \quad -\rho^{-2^j}, \quad -\rho^{-2^j}.$$

The t^{th} power of this substitution will have the same set of multipliers for $t < p^n - 1$, only if

$$t \cdot 2^j \equiv \tfrac{1}{2}(p^n - 1) - 2^j, \qquad (\text{mod } p^n - 1),$$

which may be written, when $j < k$, in the form

$$(1) \quad t \equiv 2^{-(j+1)} \cdot (p^n - 1) - 1, \qquad (\text{mod } 2^{-j}(p^n - 1)).$$

Therefore we may take

$$t = 2^{-(j+1)}(p^n - 1) - 1 + h 2^{-j}(p^n - 1),$$

or

$$t = (2h + 1)\, 2^{-(j+1)}(p^n - 1) - 1,$$

where h may take the values $0, 1, \ldots . 2^j - 1$. Hence, the substitution $A^{2^j \cdot 1}B$ is conjugate with 2^j of its powers. The particular value $j = 0$ gives the substitution B, which is therefore conjugate with one of its powers, viz. the power $\frac{1}{2}(p^n - 3)$.

When $j = k$, the congruence (1) has no solutions. It results, then, that the set of which $\{A\}$ is the typical subgroup, contains $M/2s$ conjugate cyclic subgroups. The set corresponding to the groups

$$\{A^{2^j-1}B\}, \qquad (j = 0, 1 \ldots k-1)$$

will contain $M/2^j s$ such subgroups, while the set, of which $\{A^{2^k-1}B\}$ is the typical generator, will contain M/s subgroups.

§43.—The period of the substitution A considered in the group G is the least integer r such that

$$\rho^r = \rho^{-r},$$

hence for $d = 2$ or 4, r is $\frac{1}{2}(p^n - 1)$.

The period of the substitution $\{A^{2^j-1}B\}$ is the least integer r such that

$$\rho^{2^j r} = \rho^{[\frac{1}{2}(p^n-1)-2^j]r}.$$

Hence, $$r\left(2^{j+1} - \tfrac{1}{2}(p^n - 1)\right) \equiv 0, \qquad (\text{mod. } p^n - 1).$$

or $$r\left(1 - \frac{p^n - 1}{2^{j+2}}\right) \equiv 0, \qquad \left(\text{mod } \frac{p^n - 1}{2^{j+1}}\right),$$

Hence, for $j \leqq k - 3$, $r = 2^{-(j+1)} \cdot (p^n - 1)$. When $j = k - 2$, we may write the congruence in the form

$$\tfrac{1}{2} r\left(1 - \frac{p^n - 1}{2^k}\right) \equiv 0, \qquad \left(\text{mod } \frac{p^n - 1}{2^k}\right),$$

so that $r = 2^{-k} \cdot (p^n - 1)$. When $j = k - 1$, we have

$$r\left(2 - \frac{p^n - 1}{2^k}\right) \equiv 0, \qquad \left(\text{mod } \frac{p^n - 1}{2^{k-1}}\right),$$

hence, $r = 2^{-(k-1)} \cdot (p^n - 1)$; finally, when $j = k$, we find $r = 2^{-(k-1)} \cdot (p^n - 1)$.

When $d = 2$, the order of $\{A\}$ reduces one-half, while those of $\{B\}$ and $\{AB\}$ remain unchanged in passing to G. When $d = 4$, the orders of the groups

$$\{A\}, \quad \{A^{2^j-1}B\}, \qquad (j = 0, 1, 2, \ldots . k-3)$$

are reduced one-half; the order of the group $\{A^{2^{k-2}-1}B\}$ is reduced one-fourth, while for $j = k-1$ and $j = k - 2$ the orders of the corresponding groups remain unchanged.

Since A has the multipliers $\rho, \rho, \rho^{-1}, \rho^{-1}$, $\Theta^2 A$ will have the negative of these as multipliers, and since

$$-\rho = \rho^{\frac{1}{2}(p^n+1)}, \quad -\rho^{-1} = \rho^{\frac{1}{2}(p^n-3)},$$

we see that $\Theta^2 A$ is A to the power $\frac{1}{2}(p^n-1)$.

The substitution $\Theta^2 B$ is not equal to any power of B but is conjugate with B^{-1}. Hence in G there are half as many cyclic subgroups in the set of which $\{B\}$ is the typical subgroup; the orders as we have seen remaining unchanged. The set of which $\{A\}$ is the typical subgroup has the orders of the subgroups shortened one-half, while the number of subgroups is unchanged.

The substitution $\Theta^2 AB$ is conjugate with $(AB)^{-1}$. Hence its properties in (G) are the same as those for the substitution B.

It will be shown that when $d = 4$, all these sets of cyclic groups unite to form a single set in the group G. To this end consider $\Theta A^{2^j-1} B$ whose multipliers are

$$\theta\rho^{2^j}, \quad \theta\rho^{2^j}, \quad \theta^3\rho^{-2^j}, \quad \theta^3\rho^{-2^j},$$

where $\theta = \rho^{\frac{1}{2}(p^n-1)}$. It results that

$$\Theta A^{2^j-1} B = A^t,$$

where $t = \frac{1}{4}(p^n-1) + 2^j$. This may be verified by noticing that

$$\theta\rho^{2^j} = \rho^t \quad \text{and} \quad \theta^3\rho^{-2^j} = (\theta\rho^{2^j})^{-1} = \rho^{-t}.$$

It results that all of the substitutions of the type in H are for $d = 4$ of the form A^r or ΘA^r, hence in G there is a single set of conjugate cyclic subgroups.

From §28 it follows that there are for $d = 2$ or 4, $p^n - 1 - \frac{1}{2}d$ distinct ways of setting up this canonical form for the sets of solutions there determined, are permuted in six different ways. Hence there will be $p^n - 1 - \frac{1}{2}d$ complete sets of conjugate substitutions for this type, each set containing M/s distinct substitutions.

In the group G there will be, for $d = 2$, $\frac{1}{2}(p^n-1)$ sets of conjugate substitutions; for among the $p^n - 2$ sets in H, will occur the multipliers $(1, 1, -1, -1)$ which are simply permuted if one multiplies by θ^2, while the other sets of multipliers are permuted in pairs. When $d = 4$, there is also the set $(\theta, \theta, \theta^3, \theta^3)$ which unites with the set $(-1, -1, 1, 1)$ to form a single substitution in G. The remaining $p^n - 5$ substitutions in H unite by fours; hence in all there are $\frac{1}{4}(p^n-1)$ distinct sets. In general, then, there are, for $d = 2$ or 4, $\frac{1}{d}(p^n-1)$ sets of conjugate substitutions, all but one con-

taining M/s substitutions. The exceptional set is the one arising from the multipliers $(-1, -1, 1, 1)$ when $d = 2$, and the one coming from the sets of multipliers $(-1, -1, 1, 1)$ and $(\theta, \theta, \theta^3, \theta^3)$ when $d = 4$. This set comes under the discussion of §11, the multipliers in the homogeneous group being of the form

$$M_1, \frac{1}{M_1}, \theta^2 M_1, \frac{\theta^2}{M_1},$$

Hence, there are one-half as many conjugate substitutions in G as in H, so that this set contains $M/2s$ conjugate substitutions.

Type 11 (ii).

§44.—The general substitution of this canonical form is

$$S: \quad x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \beta z, \quad w' = \beta(w + z). \qquad (\alpha^2\beta^2 = 1),$$

the period of which in H is $p(p^n - 1)$ or a factor of it, while the period of S in G is $\frac{1}{2}p(p^n - 1)$.

When $d = 1$, there are $p^{2n}(p^n - 1)$ substitutions T commutative with S, and when $d = 2$ or 4, there are $2p^{2n}(p^n - 1)$ such substitutions, viz. all of the form

$$x' = a_1 x, \quad y' = a_2 x + a_1 y, \quad z' = b_1 z, \quad w' = b_2 z + b_1 w, \qquad (a_1^2 b_1^2 = 1).$$

Just as for type 11 (i) there will be a single set of conjugate substitutions and since S and S^{2p^n-3} are the only powers of S that are conjugate, there will be $M/2p^{2n}(p^n - 1)$ substitutions in the set.

When $d = 2$ or 4, there will be an arrangement of sets of conjugate cyclic subgroups corresponding to that for the preceding type. In determining the number of cyclic subgroups in the set in H, it must be noticed that the generator is conjugate with all its powers, the exponents of which are prime to p and which have the same set of multipliers.

There will be in H, $p^n - 1 - \frac{1}{2}d$ sets of conjugate substitutions, each set containing

$$M/2p^{2n}(p^n - 1)$$

substitutions.

Type 11 (iii).

§45. $\quad S: \quad x' = \rho x, \quad y' = \rho y, \quad z' = \rho^{-1} z, \quad w' = \rho^{-1}(w + z).$

The discussion of this type is similar to that for the preceding types. The period of S in H is $p(p^n - 1)$ and in G it is $\frac{1}{2}p(p^n - 1)$.

There are

$$S = p^n(p^{2n}-1)(p^{2n}-p^n)$$

substitutions commutative with S, viz.

$x' = a_1x + b_1y,\ y' = a_2x + b_2y,\ z' = a_3z,\ w' = b_3z + a_3w,\quad (a_3[a_1b_2 - a_2b_1] = 1).$

In this case S is not conjugate with any of its powers for $d = 1$. Hence, the set contains M/s conjugate cyclic subgroups. Applying the results of §22, we see that as in type 11 (i) there are 3 sets of conjugate cyclic subgroups when $d = 2$, and $k + 3$ when $d = 4$, k being the highest power of 2 contained in $p^n - 1$.

When $d = 1$, there are $p^n - 2$ complete sets of conjugate substitutions, and when $d = 2$ or 4, $p^n - 1 - \frac{d}{2}$ sets, each set containing M/s substitutions.

§46.—As a partial verification of the correctness of the results, it will be found that the sum of the totals of the number of substitutions reducible to the various canonical forms in H is equal to M the order of H.

The following table shows the results for $p^n = 2$. The group is then of order 20160, and is simply isomorphic with the alternating group on 8 letters, On account of the low value of p^n, no substitutions occur for the types omitted from the table, and for each type present there is a single set of conjugate subgroups.

Type.	Order of Subgroups.	Number of Subgroups in set.	Number of Substitutions Reducible to Type.	Type of Corresponding Literal Substitutions.
1	15	336	4032	(12345) (12345)(678)
2	7	960	5760	(1234567)
4 (i)	6	840	1680	(12)(34)(567)
4 (ii)	3	56	112	(123)
6 (i)	3	560	1120	(123)(456)
6 (ii)	6	1680	3360	(123456)(78)
10 (i)	4	1260	2520	(1234)(56)
10 (ii)	4	630	1260	(1234)(5678)
10 (iii)	2	105	105	(12)(34)(56)(78)
10 (iv)	2	210	210	(12)(34)
Identity			1	
			20160	

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On Cardinal Numbers.

By A. N. Whitehead, *Fellow of Trinity College, Cambridge, England.*

Preface.

The following memoir is concerned solely with the theory of Cardinal numbers, and does not touch upon the theory of ordered aggregates or of ordinal types which form the other parts of the general theory of aggregates (mengenlehre).*

Sections I and II are preliminary, to explain the notation and methods of reasoning. Incidentally they may be of independent service as explaining the elements of Peano's developments of mathematical logic† (Section I) and of Russell's symbolism for the Logic of Relations‡ (Section II). I believe that these two methods are almost indispensable for the development of the theory of Cardinal numbers. The abstract nature of the subject makes ordinary language totally ineffective, only gaining precision by verbosity, and imagination is very misleading, since it presents to us special aggregates which are denumerable or of the power of the continuum. Thus we are thrown back onto a strict logical deduction by a symbolical method. I believe that the invention of the Peano and Russell symbolism, used here, forms an epoch in mathematical reasoning.

* Due almost entirely to G. Cantor. Almost all that is known respecting Cardinal Numbers, apart from their relations to the other parts of the theory, is given in his memoir in Math. Annal., Bd. XLVI, 1895: this and its sequel on Ordinal Types, loc. cit. Bd. XLIX, can be obtained as one pamphlet in a French translation, translated by F. Marotte, Paris, A. Hermann, 1899. For a general sketch of the whole theory, with bibliography, cf. "Die Entwickelung der Lehre von den Punktmannigfaltigkeiten": Bericht von A. Schönflies, "Yahresbericht der Deutschen Mathematiker-Vereinigung," 1900. For a development of the theory of Ordinal Types in respect to condensed series, by the methods of the present memoir, cf. Bertrand Russell, Revue de Mathématiques, vol. VII, Turin, 1901, where the methods of reasoning used here were first elaborated.

† For references to Peano's articles, cf. my memoir on the "Algebra of Symbolic Logic," in this Journal, vol. XXIII.

‡ Cf. Russell, loc. cit.

Section III is entirely due to Russell, and is written by him throughout. It deals with the principles of the subject, and effects a considerable simplification in them. Two classes are defined [cf. ★1·1 (all references are to the series of propositions commencing with Section III and thence numbered consecutively)] as similar when a one-one relation can be established between their numbers. The cardinal number (or power) of a class u is defined simply as the class of those classes similar to u (cf. ★1·3); thus, for instance, the cardinal number 2 is the class of those classes which (in ordinary language) contain two members, and we shall say that a class containing two members is a 2. The finite cardinals are defined [cf. ★1·4 to ★1·61] as those cardinal numbers which can be obtained by induction.* The class of infinite cardinals is defined [cf. ★1·8] as the class of cardinals which are not finite. The whole class of cardinal numbers is denoted by *Nc*. The properties of finite and infinite cardinals are deduced by strict logical reasoning from these definitions. In particular, if the cardinal number of the class of finite cardinals is denoted by α_0, it is proved that every infinite class contains a class whose cardinal number is α_0 [cf. ★2·6], and that no finite class is similar to any part of itself [cf. ★2·75], and that every infinite class is similar to some part of itself [cf. ★2·76]. The cardinal and ordinal theories of finite number are connected in ★3·3.

The known properties of α_0 are proved [★4·1 to ★4·24]. The theorems that if α and β are *any* two cardinals, and α is infinite and $\alpha \geqq \beta$, then

$$\alpha + \beta = \alpha \text{ and } \alpha \times \beta = \alpha:$$

cannot be deduced from these premises, and, as far as I am aware, have never been proved. Russell points out that the first of them follows from another unproved theorem [★4·3]. The class of numbers for which the first theorem holds is considered (cf. ★30). The second of the theorems cannot be thus deduced; but in Section V, ★20·1, it is proved that if α belong to the class for which the first theorem holds and $\beta \geqq \delta$, then

$$\beta^\alpha \times \delta^\alpha = \beta^\alpha.$$

The similar cases for $\alpha + \beta = \alpha$ are examined in ★30·0 to ★31·3.

Russell also gives exact proofs of such theorems as, $\beta < \gamma$ implies

* The recognition of the importance of Induction is due to Dedekind, cf. "Was sind and was sollen die Zahlen?" 1887.

$\alpha + \beta \leqq \alpha + \gamma$, and $\alpha \times \beta \leqq \alpha \times \gamma$, and $\alpha^\beta \leqq \alpha^\gamma$, and $\beta^\alpha \leqq \gamma^\alpha$ [cf. ★4·37, ★4·39, ★4·4, ★4·41].

Also ★4·5 contains another proof (assuming ★4·3) of Bernstein's and Schröder's theorem, that $\alpha \geqq \beta$ and $\beta \geqq \alpha$ imply $\alpha = \beta$.

Throughout Sections IV and V, I am indebted to Russell for great improvements in the symbolism of the proofs and for the correction of many oversights. Section IV is devoted to the general consideration of addition and multiplication when the number of summands or of factors is not necessarily finite or even denumerable. Cantor's definitions only apply to these cases [cf. ★7·0 to ★7·21]. The associative laws are proved [cf. ★8·1 to ★8·31, and ★10·0 to ★10·22] both for addition and multiplication when the number of brackets, as well as the number of numbers in each bracket, is not necessarily finite or denumerable. With these definitions the commutative laws become entirely irrelevant. The ordinary connection between addition and multiplication, exemplified in $\alpha + \alpha = 2 \times \alpha$, is proved [cf. ★7·3] to hold generally; also the distributive law [cf. ★11·0]. These definitions and theorems require several preliminary definitions; thus for instance [★5·1], that of a "class of classes exclusive," and [★6·0] of the "multiplicative class" of a class of classes exclusive; this last conception assumes great importance in the sequel; also, the general definitions of sums and products depend on the definition [★8·22] of "a class of classes of classes arithmetical." The remainder of Section IV is devoted to theorems following from these definitions.

Section V opens with the definition [★12·1] of a power (β^α); this definition follows from that of a product; and [★14·1] Cantor's definition is deduced from it. Thus, by this procedure, the ordinary relations between addition, multiplication, and powers of numbers are preserved in their full generality. The theory of combinations is then extended to infinite cardinals; thus, if α, β are numbers, and $\alpha \geqq \beta$, C_β^α is defined [cf. ★15·1 and ★16·0]: the extension of Vandermonde's theorem is proved [cf. ★16·1]; if α is infinite and such that $\alpha + \alpha = \alpha$, $C_\alpha^\alpha = 2^\alpha$ [cf. ★16·23]: extensions of the binomial theorem are given in ★17·0 to ★19·7; it is proved [cf. ★17·4] that if α and β are any cardinals,

$$(1 + \beta)^\alpha = \sum_{\nu \leqq \alpha} C_\nu^\alpha \times \beta^\nu.$$

If α is infinite and ν finite [cf. ★20·4], then $\nu^\alpha = 2^\alpha$, and when α is infinite and such that $\alpha^2 = \alpha$, then [cf. ★32·4], if $\beta \leqq 2^\alpha$, we have $\beta^\alpha = 2^\alpha$.

The idea of the number of permutations (P^{α}_{β}) of α things, β at a time, is extended [cf. ★22·0 and ★22·01] to the case when α and β are infinite: then [cf. ★22·1] ${}^{\beta}P_{\beta} \times C^{\alpha}_{\beta} = P^{\alpha}_{\beta}$; also, if α is infinite, and $\alpha \geqq \beta$ and with certain other limitations, then [cf. ★22·2]

$$C^{\alpha \times \beta}_{\beta} = P^{\alpha \times \beta}_{\beta} = \alpha^{\beta}.$$

Hence [cf. ★22·21], if $\alpha \times \beta = \alpha$, we have

$$C^{\alpha}_{\beta} = P^{\alpha}_{\beta} = \alpha^{\beta}.$$

In ★30·0 to ★32·6 certain classes of numbers which, without assuming ★4·3, satisfy $\beta + \gamma = \beta$ and $\beta \times \gamma = \beta$ when $\beta \geqq \gamma$ are considered.

The Italian system of numeration of propositions, on the left-hand margin, by integral and decimal numbers, is adopted. It is unnecessary for me to emphasise further, what is evident from inspection, how largely this article is based upon the ideas of G. Cantor, and of Peano, and of Russell.

SECTION I.

Account of Peano's Logical Symbolism.

Logical addition and multiplication are denoted by $\cup$ and by $\cap$ instead* of by $+$ and by $\times$; when these signs connect propositions, they indicate, as usual, that the propositions are to be taken disjunctively or conjunctively. In this memoir simple juxtaposition of terms will never be used to indicate the logical multiplication of classes; but the multiplication of propositions (which is symbolised by $\cap$) will also be indicated by separating them by one dot, thus $a \,.\, b$, or two dots, thus $a : b$, or three dots, thus $a \,.\therefore b$, or four dots, thus $a :: b$; it is to be noted that $.\therefore$ never stands for therefore.

A conjunctive or disjunctive complex of many propositions is itself one proposition. The different arrangements of dots in multiplication are therefore useful in the place of brackets.

The symbol $\supset$ between classes means "is contained in," and between propositions means "implies"; it is the same in signification as Schröder's $\preceq$. Thus, if the letters denote propositions,

$$a \supset b \,.\, x \,.\, y : \supset \,.\, u \cup v \,.\therefore \supset \,.\, w,$$

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* For example, cf. my previous memoir "On Symbolic Logic" in this Journal, vol. XXIII, No. 2.

is analysable into the hypothesis

$$a \supset b . x . y : \supset . u \cup v,$$

and into the protasis w. That symbol $\supset$, in a given chain of symbols, which is preceded or succeeded by a greater number of dots than any other symbol $\supset$ in the chain, is the main logical copula of the proposition connecting the hypothesis and the protasis: and the proposition is that the hypothesis implies the protasis.

In the above example, the hypothesis is as follows:

The proposition that the proposition a implies the proposition b, together with the propositions x and y, implies the proposition u or v or both. Thus the hypothesis is itself separable into a hypothesis and protasis. In other examples the same may be true of the protasis. In the above example, if a and b represent classes, the hypothesis becomes,

The proposition that the class a is contained in the class b, together with the propositions x and y, implies the proposition u or v or both.

The classes contained in a given class A are never to be confused with the entities which form the extension of the class A. The fact that x is a member of the class A is denoted by

$$x \in A.$$

Here $\in$ means "is a member of", or simply "is a"; $\in$ is chosen for this signification as being the initial letter of $\epsilon\sigma\tau\iota$.

Thus $y \supset A . x \in A$ may coexist with $x \in y$, but not in general with $x \supset y$ or with $x = y$, though exceptional cases arise in which $x \in y$ and $x \supset y$ coexist. As an example, let N be the class "Finite Integer", and Np the class "Prime Number", we have

$$2 \in N . Np \supset N . 2 \in Np$$

But 2 is a particular number and not a class of numbers; hence the symbolism, $2 \supset N$, is wrong.

Accordingly, $x \supset y$ implies either that x and y are both classes or that they are both propositions; and in this memoir single letters are not required to denote propositions.

If x and y are both members of the class A, then $x, y \in A$ is an abbreviation for $x \in A . y \in A$.

There are classes containing only one single member as their extension. If A is such a class, then ιA is used to denote the single member of the class A.

Also, conversely, if x is any entity, ιx is the class of which x is the single member. Hence, in symbols,

$$y \in \iota x . \supset . y = x \quad \text{and} \quad y = x . \supset . y \in \iota x,$$

and these two propositions are conveniently combined into

$$y \in \iota x . = . y = x;$$

thus the combination $\in\iota$ between classes is equivalent in signification to $=$.

For instance, the class of even primes contains the single member 2. If A denote this class, then

$$\iota 2 = A . 2 \in A . 2 = \backepsilon A : x \in A . \supset . x = 2.$$

We shall have occasion [$\star 11{\cdot}0$] to use the symbol $\iota\iota x$. This signifies the class whose single member is the class whose single member is x.

Again, if u is a class, $u \cup \iota x$ is the class whose members are the members of u together with the entity x. The symbolism $u \cup x$ for this meaning would have been wrong, for $\cup$ can only connect classes to form a new class whose extension is the joint extension of the two classes. Thus, if u and x are both classes, the two symbols $u \cup \iota x$ and $u \cup x$ have different meanings; we have

$$x \in u \cup \iota x . x \supset u \cup x;$$

also

$$y \in x . \supset . y \in u \cup x,$$

but $y \in x$ does not imply $y \in u \cup \iota x$.

The symbol elm, short for "element," stands for that class whose extension is formed by all classes whose extension is some single member. Thus, elm is a class whose members are themselves classes, but each containing only one member. Thus,

$$A \in \text{elm} . \supset : x \in A . y \in A . \supset . x = y : \supset . \backepsilon A \in A.$$

The symbol Cls, short for Class, stands for that class whose extension is formed by all classes. Thus $x \in \text{Cls}$ simply means x is a class. We get here an exception to the rule that $x \in A$ is inconsistent with $x \supset A$, for Cls is itself a class, hence,

$$\text{Cls} \in \text{Cls} . \text{Cls} = \text{Cls}.$$

If p_x denotes a proposition containing a letter x which may be given various significations consistent with the truth of the proposition, then

$$x \ni (p_x)$$

means the class whose extension is all the entities x consistent with the truth of p_x. Thus $\ni$ may be read "such that." Thus $x \ni (x^2 = 1)$ is the class whose extension is formed by ± 1, and $x \ni (x = 2)$ is the class $\iota 2$. The symbol $x \ni (p_x)$ is always to be treated as one whole: thus, if A is a class, $A \cap x \ni (p_x)$ means the class common to A and to the class $x \ni (p_x)$. Thus,

$$A \cap x \ni (p_x) . = . x \ni (x \in A . p_x).$$

This symbol is especially useful in definitions, thus, for ιx, we have

$$\iota x = y \ni (y = x),$$

and for elm we have

$$\text{elm} = \text{cls} \cap x \ni (y, z \in x . \supset . y = z).$$

If p and q are propositions containing a variable x, then

$$p . \supset_x . q$$

means that p implies q for every value of x satisfying p: this subscript to $\supset$ is not required for the main implication of a proposition, and can sometimes be omitted when it is strictly required. Similarly, if p and q are propositions containing the variables x and y, then

$$p . \supset_{x, y} . q$$

means that p implies q for every value of x and y satisfying p.

Λ is the symbol for the null class: thus,

$$A \in \text{cls} . \supset_A . \Lambda \supset A.$$

If p is a proposition, $p = \Lambda$ is the proposition which is the denial of the proposition p.

If A is a class, $\exists A$ is the proposition that "A is not Λ": thus,

$$A \in \text{cls} . \supset : \exists A . = . (A = \Lambda) = \Lambda.$$

The legitimacy of $x \in A$ implies $\exists A$; thus, from the definition $u = x \ni (p_x)$, we cannot derive the legitimacy of some proposition $x \in u$ without the premise $\exists u$.

If A is a class, $-A$ is the class of all entities which are not A; if A is a proposition, $-A$ is the proposition which is the denial of A; also $- \in$ means "is not a member of", and $- \supset$ means "is not contained in"; thus,

$$A \in \text{cls} . \supset . -A = x \ni (x - \in A) = x \ni [(x \in A) = \Lambda] = x \ni [-(x \in A)],$$

and

$$A, B \in \text{cls} . A - \supset B . \supset . \exists A - B.$$

The symbol $\cap$ between classes is replaced in this memoir by simple juxtaposition in the sole case when $\cap$ immediately precedes the symbol $-$; thus,

$$A, B \in \text{cls} . \supset . A - B = A \cap - B.$$

Also thus, if u is a class and $x \in u$, then $u - \iota x$ is the class u with the exception of the entity x. Note that $u - x$ for this meaning is wrong.

If A is a class, cls'A means the class whose members are the classes contained in A; thus,

$$A \in \text{cls} . \supset . \text{cls}'A = \text{cls} \cap x \ni (x \supset A);$$

we might also have written

$$A \in \text{cls} . \supset . \text{cls}'A = x \ni (x \supset A),$$

leaving it to be inferred from the use of $\supset$ in $x \supset A$, that $x \in \text{cls}$; thus,

$$\text{cls}'\text{cls} = \text{cls} \cap x \ni (x \supset \text{cls});$$

accordingly, cls'cls means the complete class of those classes whose members are themselves classes; thus,

$$A \in \text{cls}'\text{cls} . x \in A . \supset . x \in \text{cls}.$$

Similarly, $$\text{cls}'\text{cls}'\text{cls} = \text{cls} \cap x \ni (x \supset \text{cls}'\text{cls}),$$

and thus $$A \in \text{cls}'\text{cls}'\text{cls} . x \in A . \supset . x \in \text{cls}'\text{cls}.$$

These conceptions are essential in the general consideration of cardinal numbers. We shall abbreviate cls'cls into cls^2 and cls'cls'cls into cls^3.

We may note that

$$\text{cls}^2 \in \text{cls} \quad \text{and} \quad \text{cls}^2 \supset \text{cls};$$

also that

$$\text{cls}^3 \in \text{cls}^2, \text{ and } \text{cls}^3 \in \text{cls}, \text{ and } \text{cls}^3 \supset \text{cls}^2, \text{ and } \text{cls}^3 \supset \text{cls},$$
$$A \in \text{cls}'\text{cls} . \supset . \cup `A = x \ni [\exists A \cap P \ni (x \in P)],$$
$$A \in \text{cls}'\text{cls} . \supset . \cap `A = x \ni [P \in A . \supset_P . x \in P].$$

The class $\cup `A$ is thus the logical sum of the classes which form the extension of $A$; thus, if the extension of $A$ consists of only two classes $B$ and $C$, then $\cup `A = B \cup C$. The class $\cap `A$ is the logical product of the classes which form the extension of $A$; thus, if the extension of $A$ consists only of the two classes, $B$ and $C$, then $\cap `A = B \cap C$.

We shall, following Russell, employ frequently the following notation:

$x \in^2 A$ means that A is a cls'cls, and $x \in B$ where $B \in A$: thus in symbols,

$$A \in \text{cls}'\text{cls} . \supset : x \in^2 A . = . \exists A \cap B \ni (x \in B).$$

Note that $\exists A \frown B \ni (x \in B)$ affirms the existence of a class of B's such that $B \in A \,.\, x \in B$; it is obvious that to affirm the existence of a class of entities is the same as affirming the existence of at least one of them.

But this use of ϵ^2 brings us to Russell's "Logic of Relations", which we will now proceed to explain, shortening the explanations by the use of Peano's symbolism and numbering the definitions and propositions for reference. Df. (short for Definition) will be written after a definition, and Pp (short for Primitive proposition) after a proposition assumed without proof (cf. ★4·3 of Section III).

SECTION II.

Russell's "Logic of Relations."

Russell's* "Logic of Relations" may conveniently be considered as a generalization of the ideas suggested by Peano's proposition $x \in A$. This proposition asserts a special relation between x and A, namely, that A is a class and x is a member of its extension. Thus, according to Russell, if R is any relation, xRy signifies that x has the relation R to y. Let Rel. denote the complete class whose members are relations, then

1·00 $\qquad R \in \mathrm{Rel} \,.\, \supset : xRy \,.=.\, x$ has the relation R to y. Df.

Relations will always be denoted by capital letters to which there is a corresponding small Greek letter, and this Greek letter will have a fixed meaning, explained in 1·01, in relation to the corresponding capital italic letter. Small Greek letters will also be used in this memoir to represent cardinal numbers, but not those letters whose corresponding capital italics have been used for relations. The letters R, S will be confined to relations.

1·01 $\qquad R \in \mathrm{rel} \,.\, \supset \,.\, \rho = x \ni [\exists y \ni (xRy)]$, Df.

1·02 $\qquad R \in \mathrm{rel} \,.\, \supset \,.\, \breve{\rho} = y \ni [\exists x \ni (xRy)]$. Df.

Here $\breve{\rho}$ is read as "ρ converse". Then ρ is the class of entities such that $x \in \rho$ implies that a class of entities (and thus at least one entity) such as y exists, such that x has the relation R to y. Similarly for $\breve{\rho}$. Here ρ is called the domain of R and $\breve{\rho}$ the converse domain of R.

* Cf. Revue de Mathématiques (Turin), loc. cit.

1·03 $$R \in \text{rel} \,.\, x \in \rho \,.\supset.\, \breve{\rho}x = y \mathrel{\exists} (xRy),$$ Df.

1·04 $$R \in \text{rel} \,.\, y \in \breve{\rho} \,.\supset.\, \rho y = x \mathrel{\exists} (xRy).$$ Df.

Here $\breve{\rho}x$ and ρy are each to be considered as one symbol, and x and y are of the nature of indices.

1·05 $$u \in \text{cls} \,.\, R \in \text{rel} \,.\, u \supset \rho \,.\supset.\, \breve{\rho}u = y \mathrel{\exists} [\exists u \cap x \mathrel{\exists} (xRy)],$$ Df.

1·06 $$u \in \text{cls} \,.\, R \in \text{rel} \,.\, u \supset \breve{\rho} \,.\supset.\, \rho u = x \mathrel{\exists} [\exists u \cap y \mathrel{\exists} (xRy)],$$ Df.

1·10 $$R, R' \in \text{rel} \,.\supset.\, \therefore R \supset R' \,. \quad =: xRy \,.\supset_{x,y}.\, xR'y.$$ Df.

This defines the meaning of "the relation R is contained in the relation R'"; thus, relations have some properties analogous to those of classes.

1·11 $$R, R' \in \text{rel} \,.\supset.\, \therefore R = R' \,.=: R \supset R' \,.\, R' \supset R.$$ Df.

This defines the equivalence of relations.

1·12 $$R \in \text{rel} \,.\supset.\, \breve{R} = \imath\,\text{rel} \cap R' \mathrel{\exists} (xR'y \,.=.\, yRx),$$ Df.

$\breve{R}$ is the converse relation to R.

1·13 $$R \in \text{rel} \,.\supset.\, \breve{\breve{R}} = R,$$

1·20 $$R_1, R_2 \in \text{rel} \,.\supset: x (R_1 \cup R_2)\, y \,.=.\, xR_1y \,.\cup.\, xR_2y.$$ Df.

This defines the "logical sum" of two relations; the logical sum $R_1 \cup R_2$ is itself a relation.

1·21 $$R_1, R_2 \in \text{rel} \,.\supset: x (R_1 \cap R_2)\, y \,.=.\, xR_1y \,.\, xR_2y.$$ Df.

This defines the "logical product" of two relations; the logical product $R_1 \cap R_2$ is itself a relation.

1·22 $$R_1, R_2 \in \text{rel} \,.\supset: xR_1R_2z \,.=.\, \exists y \mathrel{\exists} (xR_1y \,.\, yR_2z).$$ Df.

This defines the "relational product" of two relations; the relational product R_1R_2 is itself a relation.

1·23 $$K \in \text{cls}\,\text{'rel} \,.\supset: x \cup \text{`}K y \,.=.\, \exists K \cap R \mathrel{\exists} (xRy).$$ Df.

$\cup$`K is a relation; it is the "logical sum" of the relations forming the extension of the class K; this definition agrees with 1·20 when the extension of K is formed by two relations only.

1·30 $$R \in \text{rel} \,.\supset.\, R^2 = RR,$$ Df.

1·31 $$R, S \in \text{rel} \,.\supset.\, \breve{}(RS) = \breve{S}\breve{R},$$

1·40 $$R \in \text{rel} \,.\supset: x - Ry \,.=.\, -(xRy).$$ Df.

Thus $-R$ is the relation of not having the relation R.

2·01 $\quad \varepsilon\varepsilon \text{ rel}, \qquad (\varepsilon \text{ is a relation}).$

2·02 $\quad x \varepsilon \breve{\varepsilon} y . = . \exists z \ni (x \varepsilon z . y \varepsilon z),$

2·03 $\quad x \breve{\varepsilon} \varepsilon y . = . \exists z \ni (z \varepsilon x . z \varepsilon y) . = . \exists xy.$

2·04 $\quad y \varepsilon \text{cls}' \text{cls} . \supset . \cup `y = x \ni (x \varepsilon^2 y),$

2·11 $\quad 1' = \text{identity}.$ Df.

1' is the relation of identity; thus, $x1'x$; and $a \varepsilon \text{elm} . x, y \varepsilon A . \supset . x1'y.$

2.12 $\quad 0' = -1'.$ Df.

Thus 0' is the relation of diversity, that is, of "being different from".

2·13 $\quad R, P \varepsilon \text{rel} . \supset : R\breve{P} \supset 0' . = . \breve{R}P \supset 0'.$

3·01 $\quad Nc \to 1 . = . \text{rel} \cap R \ni \{xRy . xRz . \supset_z . y1'z\}.$ Df.

$Nc \to 1$ is the symbol for a certain class of relations, namely, the class of "many-one" relations; any x in the domain of R has the relation R to only one member of the converse domain.

3·02 $\quad 1 \to Nc . = . \text{rel} \cap R \ni \{yRx . zRx . \supset x . y1'z\}.$ Df.

Thus $1 \to Nc$ is the class of one-many relations.

3·03 $\quad R \varepsilon Nc \to 1 . = . \breve{R} \varepsilon 1 \to Nc,$

3·04 $\quad R \varepsilon 1 \to Nc . \supset . R\breve{R} \supset 1',$

that is, $\quad xRy . y\breve{R}z . \supset . x1'z.$

3·11 $\quad 1 \to 1 . = . (Nc \to 1) \cap (1 \to Nc).$

$1 \to 1$ is the class of one-one relations.

3·12 $\quad R \varepsilon 1 \to 1 . \supset . \breve{R} \varepsilon 1 \to 1.$

3·13 $\quad R, S \varepsilon 1 \to 1 . \supset . RS \varepsilon 1 \to 1,$

4·11 $\quad S \varepsilon Nc \to 1 . R = S\breve{S} . \supset . R^2 \supset R . R = \breve{R}.$

4·12 $\quad R \varepsilon \text{rel} . R^2 \supset R . R = \breve{R} . \exists R . \supset . \exists Nc \to 1 \cap S \ni (R = S\breve{S}).$

Note that $\quad R^2 \supset R . = : xRy . yRz . \supset . xRz,$

and that $\quad R = \breve{R} . = : xRy . = . yRx.$

5·11 $\quad R \varepsilon \text{rel} . u \supset \rho . \supset : xR_u y . = . x \varepsilon u . xRy.$ Df.

5·12 $\quad R \varepsilon \text{rel} . u \supset \rho . \supset . \rho_u = u . \breve{\rho}_u = \breve{\rho}u.$

R_u is the relation R with its field limited, namely, the domain is limited to u, and thus the converse domain is limited to $\breve{\rho}u$.

For the rest of this memoir the propositions are numbered progressively onwards, and all references to propositions without indication of section refer to this series of numbers. In the future it will occur that a general hypothesis applies to a number of propositions: such a hypothesis will be stated only once, after mention of the integral number belonging to that set of propositions, and will be understood to hold till the integral number is changed. Demonstrations are placed in square brackets after the enunciations; subsidiary propositions used only in the demonstrations are numbered on the right instead of on the left, and only integral numbers in brackets are used for such numeration; thus, (2) would refer to a subsidiary proposition used in that demonstration and ★3·11 would refer to a proposition in the main body of the article; thus, (1).(2).★3·11.⊃. means that "the subsidiary propositions (1) and (2), together with the main proposition 3·11 imply" the protasis. The hypothesis of the proposition to be proved is cited by "Hp"; thus, Hp.(1) means "the hypothesis together with the subsidiary proposition (1)"; but Hp (1) means the hypothesis of the subsidiary proposition (1) and Hp (3·11) the hypothesis of the previous proposition ★3·11.

SECTION III.

On Finite and Infinite Cardinal Numbers.

This section III is entirely due to Russell, with the exception of some of the notes.

★1

·1 $$u \operatorname{sim} v .=. u, v \in \operatorname{cls} . \exists 1 \to 1 \cap R \ni (u \supset \rho . v = \breve{\rho} u).$$ Df.

Note: sim is contracted from similar, and u sim v can be read: u is similar to v. Thus sim is a symbol for a relation, and has all the properties ascribed to such symbols in Section II.

·2 $$u \in \operatorname{cls} . \supset . \mu u = \operatorname{cls} \cap v \ni (u \operatorname{sim} v).$$ Df.

·3 $$Nc = \operatorname{cls}^2 \cap z \ni \{\exists \operatorname{cls} \cap u \ni (z = \mu u)\}$$ Df.

·4 $$0 = \iota \Lambda .$$ Df.

·41 $$1 = \operatorname{cls} \cap u \ni (x \in u . \supset . u - \iota x \in 0).$$ Df.

Note: 1 has the same signification as elm (cf. Section I) and will now replace that symbol.

$\star 1{\cdot}42$ $2 = \mathrm{cls} \cap u \ni (x \in u \,.\, \supset \,.\, u - \iota x \in 1)$. Df.

$\cdot 43$ $3 = \mathrm{cls} \cap u \ni (x \in u \,.\, \supset \,.\, u - \iota x \in 2)$. Df.

$\cdot 5$ $n \in Nc \,.\supset.\, n+1 - \mathrm{cls} \cap u \in (x \in u \,.\, \supset \,.\, u - \iota x \in n)$. Df.

$\cdot 6$ $Nc \,\mathrm{fin} = Nc \cap n \ni \{s \in \mathrm{cls} \,.\, 0 \in s : m \in Nc \cap s \,.\supset_m.\, m+1 \in s : \supset \,.\, n \in s\}$. Df.

$\cdot 61$ $\mathrm{Induct} \,. = .\therefore s \in \mathrm{cls} \,.\, 0 \in s : m \in Nc \cap s \,.\supset_m.\, m+1 \in s : \supset_s .\, Nc \,\mathrm{fin} \supset s$ Df.

Note: Induct is contracted from Induction.

$\cdot 7$ $0 \in Nc \,\mathrm{fin}$.

$\cdot 71$ $1 \in Nc \,\mathrm{fin}$, [$1 = 0+1 \,.\, 0 \in Nc \,\mathrm{fin} \,.\supset.$ Prop].

$\cdot 72$ $2 \in Nc \,\mathrm{fin}$.

$\cdot 73$ $n \in Nc \,\mathrm{fin} \,.\supset.\, n+1 \in Nc \,\mathrm{fin}$, [Induct].

$\cdot 8$ $Nc \,\mathrm{infin} = Nc - Nc \,\mathrm{fin}$. Df.

1.81 $\mathrm{cls} \,\mathrm{fin} = \cup \,`Nc \,\mathrm{fin}$. Df.

$\cdot 82$ $\mathrm{cls} \,\mathrm{infin} = \cup \,`Nc \,\mathrm{infin}$. Df.

$.83$ $\mathrm{cls} \quad = \mathrm{cls} \,\mathrm{fin} \cup \mathrm{cls} \,\mathrm{infin}$.

$\cdot 84$ $\mathrm{cls} \,\mathrm{fin} \cap \mathrm{cls} \,\mathrm{infin} = \Lambda$.

$\cdot 91$ $u \,\mathrm{sim}\, u' \,.\, v \,\mathrm{sim}\, v' \,.\, u \cap v = \Lambda \,.\, u' \cap v' = \Lambda \,.\supset.\, u \cup v \,\mathrm{sim}\, u' \cup v'$,
[$S,\ S' \in 1 \to 1 \,.\, u = \sigma \,.\, u' = \breve{\sigma} \,.\, v = \sigma' \,.\, v' = \breve{\sigma}' .\ S \cup S' = S'' \,.\supset.$
$S'' \in 1 \to 1 \,.\, \sigma'' \in u \cup v \,.\, \breve{\sigma}'' = u' \cup v' \,.\supset.$ Prop].

$\cdot 92$ $\alpha, \beta \in Nc \,.\supset.\, \alpha + \beta = \imath' Nc \cap z \ni \{u \in \alpha \,.\, v \in \beta \,.\, u \cap v = \Lambda \,.\supset.\, u \cup v \in z\}$. Df.

$\cdot 93$ $m \in Nc \,.\, n \in Nc \,\mathrm{fin} \,.\supset.\, m + (n+1) = \imath' Nc \cap z \ni \{u \in m \,.\, v \in n \,.$
$u \cap v = \Lambda \,.\, x - \in u \cup v \,.\, v \cup \iota x = v' \,.\supset.\, u \cup v' \in z\}$.

$\cdot 94$ $(m+n)+1 = \imath' Nc \cap z \ni \{w \in m+n \,.\, x - \in w \,.\supset.\, w \cup \iota x \in z\}$.

$\cdot 95$ $m + (n+1) = (m+n) + 1$.

$\star 2$

$\cdot 1$ $n \in Nc \,\mathrm{fin} \,.\supset.\, n - = n+1$,
[$0 - = 1 \ldots .$ (1) $n - = n+1 \,. = : u \in n \,.\, x - \in u \,.\supset.\, u \cup \iota x - \mathrm{sim}\, u :$
$= : u \cup \iota x \in n+1 \,.\, y - \in u \cup \iota x \,.\supset.$
$u \cup \iota x \cup \iota y - \mathrm{sim}\, u \cup \iota x :$
$= : n+1 - = (n+1)+1,$ (2)
(1). (2). Induct .$\supset$. Prop].

$\cdot 2$ $R = \imath' 1 \to 1 \cap R' \ni (\rho' = Nc \,\mathrm{fin} : xR'y \,. = . \,x+1 = y)$. Df.

Note: This special signification for R, and thus for ρ and $\breve{\rho}$, is only retained throughout Section III.

$\star 2{\cdot}3$ $R \supset 0'$, [$\star 2{\cdot}1 . \supset .$ Prop].

$\cdot 31$ $\breve{\rho} \supset \rho . \exists \rho - \breve{\rho}$, [$\breve{\rho} = \rho - \iota 0$].

$\cdot 32$ Induct $. = : s \in \text{cls} . 0 \in s . \breve{\rho}(s \cap \rho) \supset s . \supset_s . \rho \supset s$.

$\cdot 4$ $\alpha_0 = \text{cls} \cap s \ni \{\exists 1 \to 1 \cap S \ni (\breve{\sigma}) \supset \sigma . \exists \sigma - \breve{\sigma} : s \supset \sigma . \breve{\sigma} s \supset s . \exists s - \breve{\sigma} s . \supset . s = \sigma)\}$. Df.

$\cdot 41$ $\alpha_0 \in Nc$, [cf. Russell, Révue de Math., loc. cit., §3, Prop. 1·9].

$\cdot 42$ $Nc \text{ fin} \in \alpha_0$, [$\star 2{\cdot}31 . \star 2{\cdot}32 . \supset .$ Prop].

$\cdot 43$ $\alpha_0 \in Nc \text{ infin}$, [$\alpha_0 = \mu Nc \text{ fin} . Nc \text{ fin} = \rho . \mu\rho = \mu\breve{\rho} = \mu\breve{\rho} + 1 . \star 2{\cdot}1 . \supset .$ Prop].

$\cdot 5$ $n \in Nc - \iota 0 . \supset . n - 1 = \imath' Nc \cap x \ni (x + 1 = n)$. Df.

$\cdot 51$ $n \in Nc \text{ infin} . \supset . n - 1 \in Nc \text{ infin}$, [$\star 1{\cdot}8 . \star 1{\cdot}73 . \supset .$ Prop].

$\cdot 6$ $n \in Nc \text{ infin} . u \in n . \supset . \exists \alpha_0 \cap \text{cls}'u$,

[$u \in n . x \in u . \star 2{\cdot}51 . \supset . u - \iota x \in \text{cls infin}$, (1)

$m \in Nc \text{ fin} . u_m \in m . u_m \supset u . u - u_m \in \text{cls infin} . x \in u - u_m . \supset . u - u_m - \iota x \in \text{cls infin}$, (2)

(1) . (2) . Induct $. \supset : m \in Nc \text{ fin} . u_m \in m . u_m \supset u . \supset . u - u_m \in \text{cls infin}$, (3)

$K = \text{cls}'\text{cls} \cap v \ni \{v \supset \text{cls}'u \cap \text{cls fin} : : m \in Nc \text{ fin} . \supset_m . \therefore v \cap m \ni 1 : p, p' \in v .$
$p \in m . p' \in m + 1 . \supset . p \supset p'\} . v \in K . \supset . \cup' v \in \alpha_0 . \cup' v \supset u$. (4)

(3) $\supset \exists K \ldots .$ (5), (4) . (5) . $\supset .$ Prop].

$\cdot 7$ $u \in \text{cls infin} . v \in \text{cls fin} . v \supset u . \supset . u \text{ sim } u - v$,

[$\star 2{\cdot}6 . \supset . \exists \alpha_0 \cap \text{cls}'u$, (1)

$w \in \alpha_0 \cap \text{cls}'u . v' \in \text{cls fin} . v' \supset w . \supset . w \text{ sim } w - v' . \supset . u \text{ sim } u - v'$, (2)

$v' \text{ sim } v . \supset . u - v' \text{ sim } u - v \ldots .$ (3), (2) . (3) . $\supset .$ Prop].

$\cdot 71$ $\alpha \in Nc \text{ infin} . n \in Nc \text{ fin} . \supset . \alpha - n = \alpha = \alpha + n$, [$\star 2{\cdot}7 . \supset .$ Prop].

$\cdot 72$ $m, n \in Nc \text{ fin} . \supset . m + n - = m$, [$\star 2{\cdot}4 . \supset .$ Prop].

$\cdot 73$ $m + n \in Nc \text{ fin} . \supset . m, n \in Nc \text{ fin}$.

$\cdot 74$ $m, n \in Nc . \supset : m + n = m . = . m \in Nc \text{ infin}$.

$\cdot 75$ $Nc \text{ fin} = Nc \cap n \ni (n - = n + 1)$, [$\star 2{\cdot}1 . \star 2{\cdot}71 . \text{Transp} . \supset .$ Prop].

$\cdot 76$ $Nc \text{ infin} = Nc \cap \alpha \ni (\alpha = \alpha - 1)$, [$\star 2{\cdot}1 . \text{Transp} . \star 2{\cdot}71 . \supset .$ Prop].

The Props $\star 2{\cdot}75$ and $\star 2{\cdot}76$ show that from the Df of finite numbers by induction we can deduce the properties of infinity with regard to similarity of whole and part.

$\star 3$

$\cdot 1$ $\alpha, \beta \in Nc . \supset : \alpha < \beta . = . \alpha \cap \beta = \Lambda . \exists \alpha \cap u \ni \{\exists \beta \cap v \ni (u \supset v)\}$. Df.

$\cdot 12$ $\alpha, \beta \in Nc . \supset : \alpha > \beta . = . \beta < \alpha$, Df.

$\cdot 12$ $\alpha, \beta \in Nc . \supset : \alpha \leqq \beta . = . \exists \alpha \cap u \ni \{\exists \beta \cap v \ni (u \supset v)\}$. Df.

$\star 3{\cdot}2$ $\alpha, \beta \in Nc . \alpha < \beta . v \in \beta . \supset . \exists \alpha \cap \text{cls}'v$

$[u_0 \in \alpha . v_0 \in \beta . u_0 \supset v_0 . v \in \beta . \supset . \exists 1 \to 1 \cap S \ni (\sigma = v_0 . \breve{\sigma} = v . \breve{\sigma} u_0 \in \alpha \cap \text{cls}'v)]$.

$\cdot 3$ $m, n \in Nc \text{ fin} . \supset : m \leqq n . = . m R^{n-m} n$, [cf. $\star 2{\cdot}2$],

$[a, b \in Nc \text{ fin} . aRb . \supset . \exists a \cap u \ni (x - \in u . \supset . u \cup \iota x \in b) . \supset : v \in b .$

$\supset_v . \exists a \cap \text{cls}'v$, (1)

$a, b, c \in Nc \text{ fin} . aR^c b . v \in b . \exists a \cap \text{cls}'v . \supset : x - \in v . \supset . \exists a \cap \text{cls}'(v \cup \iota x)$. (2)

(1). (2). Induct. $\supset : a, b, c \in Nc \text{ fin} . aR^c b . v \in b . \supset . \exists a \cap \text{cls}'v$, (3)

$aRb . \supset . a = b - 1 . 1 = b - a$. (4)

$aR^c b . \supset . c = b - a : \supset : aR^{c'}(b+1) . \supset . c' = b + 1 - a$, (5)

(4). (5). Induct. $\supset : aR^c b . \supset . c = b - a$, (6)

(3). (6). $\supset : a, b, c \in Nc \text{ fin} . aR^{b-a} b . \supset . a \leqq b$, (7)

$a = b . \supset . aR^0 b . \supset . aR^{b-a} b$, (8)

$a < b . u \in a . v \in b . u \supset v . v - u \in c . \supset . aR^c b . c = b - a$, (9)

(7). (8). (9). $\supset$. Prop].

Note: This prop. connects the cardinal and ordinal theories of finite number.

$\star 4$

$\cdot 1$ $\alpha \in Nc \text{ infin} . \supset . \alpha \geqq \alpha_0$. [$\star 2{\cdot}6 . \supset$. Prop].

$\cdot 2$ $\alpha_0 = \alpha_0 + \alpha_0$,

$[R' = \imath \text{Rel} \cap P \ni \{xPy . = . \exists Nc \text{ fin} \cap n \ni (xR^{2n}y)\} .$

$\rho_0 = \breve{\rho}'0 . \rho_1 = \breve{\rho}'1 . \supset . \rho_0 \cap \rho_1 = \Lambda . \rho_0 \cup \rho_1 = \rho . \rho_0, \rho_1 \in \alpha_0 . \supset$. Prop].

$\cdot 21$ $n \in Nc \text{ fin} . \supset . \alpha_0 = \alpha_0 \times n$, [$\star 4{\cdot}2$. Induct. $\supset$. Prop].

$\cdot 22$ $\alpha_0 = \alpha_0 \times \alpha_0$,

$[M = x \ni \{\exists Nc \text{ fin} \cap (m, n) \ni (x = \iota m \cup \iota n)\} . \supset . \mu M = \alpha_0 \times \alpha_0$, (1)

$R' \in 1 \to 1 . \rho' = M . \therefore \iota m \cup \iota n R' \iota m' \cup \iota n' . = : m > n . m - n > 1 .$

$m' + n' = m + n . m' = m - 1 . \cup . m \geqq n . m - n \leqq 1 .$

$m' + n' = m + n + 1 . n' = 0 . \therefore \supset . \rho' \in \alpha_0 . \supset$. Prop].

$\cdot 23$ $n \in Nc \text{ fin} . \supset . \alpha_0 = \alpha_0^n$, [Induct].

$\cdot 24$ $u, v \in \text{cls}'Nc \text{ fin} . \supset . \sum_{x \in u . y \in v} x \times \alpha^y = \alpha_0$.

$\cdot 3$ $(u \in \text{cls infin} . \supset . \exists \text{ cls}'\alpha_0 \text{ excl} \cap v \ni (\cup `v = u)$. Pp.

For definition of $\text{cls}'\alpha_0$ excl, cf $\star 5{\cdot}1$ later.)

Note: It has been shown that every infinite class contains classes whose number is α_0. Thus if a class v of α_0's has been found which contains all but a finite number of u's, these may be added on to the v's; but if an infinite number of u's remain, new α_0's may be found out of them, and to this process there can

be no end as long as any u's remain. It appears not impossible that a theorem may be discovered proving that this process can always be completed. In any case, it is much to be desired that the possibility in all cases of this completion of the above process should be either proved or disproved. In order to show the importance of this investigation, the theorem is here hypothetically assumed and stated as a primitive proposition (★4·3), and some immediate deductions (★4·31 to ★4·51) are made from it. The subject is resumed in the sets of propositions ★30, ★31, and ★32.

$\alpha \in Nc$ infin . ★4·3 . ⊃ :

★4·31 $\vdash Nc \cap \beta \ni (\alpha = \alpha_0\beta)$, [★4·3 . ⊃ . Prop].

·32 $\alpha + \alpha = \alpha$, $[\alpha = \alpha_0\beta = (\alpha_0 + \alpha_0)\,\beta = \alpha + \alpha]$.

·33 $\alpha\alpha_0 = \alpha$, $[\alpha = \alpha_0\beta = \alpha_0^2\beta = \alpha\alpha_0]$.

·34 $\alpha \in Nc \cap \beta \ni (\alpha = \alpha_0\beta)$, [★4·33 . ⊃ . Prop].

·35 $\alpha > \alpha_0 . \beta < \alpha . \supset . \alpha_0\beta < \alpha$,

$[\beta \in Nc$ fin . ⊃ . $\alpha_0\beta = \alpha_0 . \supset . \alpha_0\beta < \alpha$, (1)

$\beta \in Nc$ infin . ⊃ . $\alpha_0\beta = \beta . \supset . \alpha_0\beta < \alpha$, (2)

(1) . (2) . ⊃ . Prop].

·36 $\alpha > \alpha_0 . \supset . Nc \cap \beta \ni (\alpha = \alpha_0\beta) = \iota\alpha$, [★4·35 . ⊃ . Prop].

·37 $\beta < \gamma . \supset . \alpha + \beta \leqq \alpha + \gamma$,

$[u \in \alpha . v \in \beta . w \in \gamma . v \supset w . u \cap w = \Lambda . \supset . u \cup v \supset u \cup w . \supset .$ Prop].

·38 $\beta < \alpha . \supset . \alpha + \beta = \alpha$, $[\alpha \leqq \alpha + \beta \leqq \alpha + \alpha . \alpha = \alpha + \alpha . \supset .$ Prop].

·39 $\beta < \gamma . \supset . \alpha\beta \leqq \alpha\gamma$,

$[v \in \beta . w \in \gamma . v \supset w . w \supset \alpha . \supset . \cup\, `v \supset \cup\, `w . \supset .$ Prop].

·4 $\beta < \gamma . \supset . \alpha^\beta \leqq \alpha^\gamma$,

$[u \in \alpha . v \in \beta . w \in \gamma . v \supset w . \supset . \alpha^\beta = \mu Nc \to 1 \cap S \ni (\sigma = v . \breve{\sigma} \supset u) . \supset .$

$\alpha^\gamma = \mu Nc \to 1 \cap S \ni (\sigma = w . \breve{\sigma} \supset u) . \supset .$ Prop].

Note: Here Cantor's definition of the power of a cardinal number (e. g. α^β) is used, cf. ★14·1 of Section V.

·41 $\beta < \gamma . \supset . \beta^\alpha \leqq \gamma^\alpha$,

$[u \in \alpha . v \in \beta . w \in \gamma . v \supset w . \supset . Nc \to 1 \cap S \ni (\sigma = u . \breve{\sigma} \supset v) \in \beta^\alpha .$

$Nc \to 1 \cap S \ni (\sigma = u . \breve{\sigma} \supset w) \in \gamma^\alpha$, (1)

(1) . $Nc \to 1 \cap S \ni (\sigma = u . \breve{\sigma} \supset v) \supset Nc \to 1 \cap S \ni (\sigma = u . \breve{\sigma} \supset w) . \supset .$ Prop].

·42 $n \in Nc$ fin . ⊃ . $n\alpha = \alpha . \alpha\alpha_0^n = \alpha$.

·43 $\beta, \gamma \in Nc . \beta \leqq \alpha . \supset . \alpha^\gamma = (\alpha + \beta)^\gamma$.

·44 $\beta \in Nc . \gamma \in Nc$ infin . $\beta \leqq \alpha . \supset . \alpha^\gamma\beta^\gamma = \alpha^\gamma$.

[This theorem is enunciated and proved in Section V, ★20·1; it is mentioned here for the sake of completeness.]

★4·5 $u, v \in \mathrm{cls} \,.\, \exists \mu u \cap \mathrm{cls}'v \,.\, \exists \mu v \cap \mathrm{cls}'u \,.\supset.\, u \,\mathrm{sim}\, v.$

Note: This is Schröder's and Bernstein's theorem; cf. Borel, "Théorie des Functions," Paris, 1898, p. 108 ff.

$[S \in 1 \to 1 \,.\, \sigma = u \,.\, \breve{\sigma} \supset v \,.\, S' \in 1 \to 1 \,.\, \sigma' = v \,.\, \breve{\sigma}' \supset u \,.\supset.\, \sigma(\sigma') = u \,.\, \breve{\sigma}'(\breve{\sigma}) \supset \breve{\sigma}' \supset u,$ (1)

$(1) \,.\, \mu u = \alpha \,.\, \mu v = \beta \,.\, \mu u - \breve{\sigma}' = \gamma \,.\, \mu \breve{\sigma}' - \breve{\sigma}'(\breve{\sigma}) = \delta \,.\supset.$

$\alpha \geq \beta \,.\, \beta \geq \alpha \,.\, \alpha = \beta + \gamma \,.\, \beta = \alpha + \delta \,.\supset.\, \alpha = \alpha + \gamma + \delta,$ (2)

$(2) \,.\, ★4·38 \,.\supset.\, \gamma + \delta \leq \alpha \,.\supset.\, \delta \leq \alpha \,.\supset.\, \alpha + \delta = \alpha \,.\supset.\, \beta = \alpha \,.\supset.\, \mathrm{Prop}].$

Note: Bernstein's proof of Schröder's and Bernstein's theorem given by Borel (loc. cit.) does not depend upon ★4·3.

·51 $\alpha, \beta \in Nc \,.\, \alpha > \beta \,.\supset.\, - (\beta \geq \alpha),$ $[★4·5 \,.\supset.\, \mathrm{Prop}].$

SECTION IV.

★5

·1 $\mathrm{cls}^2\,\mathrm{excl} \,.=.\, \mathrm{cls}^2 \cap d \ni \{q, q' \in d \,.\supset.\, q = q' \,.\cup.\, q \cap q' = \Lambda\}.$ Df.

Note: excl is contracted from "exclusive".

·2 $a \in \mathrm{cls}^2 \,.\supset.\, \mathrm{cls}'a\,\mathrm{excl} = \mathrm{cls}'a \cap \mathrm{cls}^2\,\mathrm{excl}.$ Df.

★6 $d \in \mathrm{cls}^2\,\mathrm{excl} \,.\supset.\, \dot{}.$

·0 $d^{\times} = \mathrm{cls} \cap m \ni \{p \in d \,.\supset_p .\, p \cap m \in 1 : m \supset \cup `d\}.$ Df.

Note: $d^{\times}$ is the "multiplicative" class of d, and can be read as "d multiplicative": it has no existence, according to this definition, unless d is a cls^2 excl.

·11 $p \in d \,.\, x \in p \,.\, d_1 = d - \iota p \,.\, m_1 \in d_1^{\times} \,.\, m = m_1 \cup \iota x \,.\supset.\, m \in d^{\times},$

$[\mathrm{Hp} \,.\, p' \in d \,.\, p' - = p \,.\supset.\, p' \in d_1 \,.\, x - \in p' \,.\supset.\, m \cap p' = m_1 \cap p' \,.\supset.\, m \cap p' \in 1.$ (1)

$\mathrm{Hp} \,.\supset.\, m \cap p = \iota x \,.\supset.\, m \cap p \in 1.$ (2)

$(1) \,.\, (2) \,.\supset: p'' \in d \,.\supset.\, p'' \cap m \in 1 \ldots .$ (3) $m \supset \cup `d,$ (4)

$(3) \,.\, (4) \,.\supset.\, \mathrm{Prop}].$

·12 $u \supset \cup `d : x, y \in u \,.\, x 0' y \,.\supset_{x,y} .\, - \exists d \cap p \ni (x, y \in p) : \supset \,.\, \exists d^{\times} \cap m \ni (u \supset m),$

$[\mathrm{Hp} \,.\, p \in d \,.\, \exists p \cap u \,.\supset.\, p \cap u \in 1,$ (1)

$d_u = d \cap p \ni (p \cap u = \Lambda) \,.\supset.\, d_u \in \mathrm{cls}^2\,\mathrm{excl} \,.\supset.\, \exists d_u^{\times},$ (2)

$(1) \,.\, m_1 \in d_u^{\times} \,.\, m = m_1 \cup u \,.\supset: p \in d \,.\supset_p .\, p \cap m \in 1,$ (3)

$m \supset \cup `d \ldots .$ (4), $(3) \,.\, (4) \,.\supset.\, m \in d^{\times} \,.\supset.\, \mathrm{Prop}].$

·13 $m \in d^{\times} \,.\supset.\, - \exists d^{\times} \cap m' \ni (m' o' m \,.\, m' \supset m),$

$[m' o' m \,.\, m' \supset m \,.\supset.\, \exists d \cap p \ni (m' \cap p = \Lambda) \,.\supset.\, m' - \in d^{\times} \,.\supset.\, \mathrm{Prop}].$

$\star 6{\cdot}14$ $l \varepsilon \operatorname{cls} . \supset . \therefore d^{\times} \cap \operatorname{cls}' l \varepsilon 1 . = : p \varepsilon d . \supset_p . p \cap l \varepsilon 1,$

$[\star 6{\cdot}0 . \operatorname{Hp} . \supset : p \varepsilon d . \supset_p . \exists p \cap l,$ (1)

$m = \imath d^{\times} \cap \operatorname{cls}' l . p \varepsilon d . x = \imath p \cap m . \supset . x \varepsilon l,$ (2)

$\operatorname{Hp}(2) . y \varepsilon p \cap l . \supset . m - \iota x \cup \iota y \varepsilon d^{\times} \cap \operatorname{cls}' l . \supset . m - \iota x \cup \iota y = m . \supset . x 1' y,$ (3)

$(1) . (2) . (3) . \supset . \therefore d^{\times} \cap \operatorname{cls}' l \varepsilon 1 . \supset : p \varepsilon d . \supset_p . p \cap l \varepsilon 1$ (4)

$\star 6{\cdot}0 . \supset . \therefore p \varepsilon d . \supset_p . p \cap l \varepsilon 1 : \supset . d^{\times} \cap \operatorname{cls}' l \varepsilon 1,$ (5)

$(4) . (5) . \supset . \operatorname{Prop}].$

$\cdot 21$ $\cup `d^{\times} = \cup `d.$

$\cdot 22$ $f \varepsilon \operatorname{cls}^2 \operatorname{excl} . \supset : d^{\times} = f^{\times} . = . d = f,$

$[\star 6{\cdot}21 . \operatorname{Hp} . \supset . \cup `d = \cup `f,$ (1)

$p \varepsilon d . x, y \varepsilon p . \supset . - \exists d^{\times} \cap m \ni (x, y \varepsilon m) . \supset . - \exists f^{\times} \cap m \ni (x, y \varepsilon m),$ (2)

$x, y \varepsilon^2 f . - \exists f^{\times} \cap m \ni (x, y \varepsilon m) . \supset . \exists f \cap q \ni (x, y \varepsilon q).$ (3)

$(1) . (2) . (3) . \supset : d^{\times} = f^{\times} . \supset . d = f,$ (4)

$(4) : d = f . \supset . d^{\times} = f^{\times} : \supset . \operatorname{Prop}].$

$\star 7$

$\cdot 0$ $R \varepsilon 1 \to 1 . R \supset \operatorname{sim} . \rho, \breve{\rho} \varepsilon \operatorname{cls}^2 \operatorname{excl} . \supset . \cup `\rho \operatorname{sim} \cup `\breve{\rho},$

$[u \varepsilon \rho . \supset . \exists 1 \to 1 \cap R' \ni (\rho' = u . \breve{\rho}' = \imath \breve{\rho} u),$ (1)

$K = x \ni [\exists \rho \cap u \ni \{x = 1 \to 1 \cap R' \ni (\rho' = u . \breve{\rho}' = \imath \breve{\rho} u)\}] .$

$L \varepsilon K^{\times} . S = \cup `L . \supset . S \varepsilon 1 \to 1 . \sigma = \cup `\rho . \breve{\sigma} = \cup `\breve{\rho} . \supset . \operatorname{Prop}].$

This prop proves that if d is a $\operatorname{cls}^2 \operatorname{excl}$, $\mu \cup `d$ is defined entirely by the numbers involved in d and its classes; thus we get the conception of a sum of numbers.

$\cdot 1$ $d \varepsilon \operatorname{cls}^2 \operatorname{excl} . \supset . \sum_{p \varepsilon d} \mu p = \mu \cup `d.$ Df.

Thus $\sum_{p \varepsilon d} \mu p$ is the sum of all the numbers of the classes forming the extension of the class d. The usual notation $\alpha + \beta + \gamma + \ldots$ is only suitable for a finite, or at least for a well-ordered set of numbers; also, it introduces unnecessarily the idea of order among the numbers added, and thus renders necessary the totally irrelevant commutative law. But we shall use it, when convenient (e. g. Section III above), for a finite number of numbers; thus,

$$\alpha, \beta \varepsilon Nc . p \varepsilon \alpha . q \varepsilon \beta . d = \iota p \cup \iota q . \supset . \sum_{u \varepsilon d} \mu u = \alpha + \beta = \beta + \alpha = \mu \cup `d.$$

$\cdot 2$ $R \varepsilon 1 \to 1 . R \supset \operatorname{sim} . \rho, \breve{\rho} \varepsilon \operatorname{cls}^2 \operatorname{excl} . \supset . \rho^{\times} \operatorname{sim} \breve{\rho}^{\times},$

$[u \varepsilon \rho . \supset . \exists 1 \to 1 \cap R' \ni (\rho' = u . \breve{\rho}' = \imath \breve{\rho} u).$ (1)

$K = x \ni [\exists \rho \cap u \ni \{x = 1 \to 1 \cap R' \ni (\rho' = u . \breve{\rho}' = \imath \breve{\rho} u)\}] . L \varepsilon K^{\times} .$

$S = \cup `L . \supset . \therefore m \varepsilon \rho^{\times} . \supset_m . \breve{\rho}^{\times} \cap n \ni (n = \breve{\sigma} m) \varepsilon 1 : n \varepsilon \breve{\rho}^{\times} . \supset_n .$

$\rho^{\times} \cap m \ni (m = \sigma n) \varepsilon 1 . \therefore \supset . \operatorname{Prop}].$

Analogous remarks to those on ★7·0 apply here; we derive from this prop the conception of a product of numbers.

★7·21 $$d \in \text{cls}^2 \text{ excl} . \supset . \prod^{p \in d} \mu p = \mu d^{\times}. \quad \text{Df.}$$

Thus, $\prod^{p \in d} \mu p$ is the product of all the numbers of the classes forming the extension of the class d. Analogous remarks to those on ★7·1 apply to the notation $\alpha \times \beta \times \gamma \ldots$. for a product of cardinal numbers.

·3 $$\alpha, \beta \in Nc . d \in \alpha \cap \text{cls}'\beta \text{ excl} . \supset . \cup `d \in \alpha \times \beta,$$

$$[d \supset \beta . \supset : p, p' \in d . \supset . \exists 1 \to 1 \cap R \ni (p' = \rho . p = \breve{\rho}), \quad (1)$$

$$p \in d . \supset . R(p) = x \ni \{\exists d \cap p' \ni [x = 1 \to 1 \cap R \ni (p' = \rho . p = \breve{\rho})]\} :$$

$$P \in R(p)^{\times} . S = \cup `P . k = x \ni \{\exists p \cap y \ni (x = \sigma y)\} : \supset . \therefore \sigma = \cup `d .$$

$$\breve{\sigma} = p . k \in \beta \cap \text{cls}'\alpha \text{ excl} . k \supset d^{\times}, \quad (3)$$

$$(3) . h = \iota d \cup \iota k . \supset . h^{\times} \in \alpha \times \beta, \quad (4)$$

$$k \supset d^{\times} . x \in h^{\times} . \supset . \exists d \cap p' \ni \{\exists k \cap q \ni (x = \iota p' \cup \iota q .$$

$$p' \cap q \in 1 . \imath p' \cap q \in \cup `d)\}, \quad (5)$$

$$y \in \cup `d . \supset . h^{\times} \cap x \ni (\cap `x = \iota y) \in 1, \quad (6)$$

$$(4) . (5) . (6) . \supset . \text{Prop}].$$

Note: The proposition connects addition with the multiplication of two numbers. It also gives a signification to the commutative law $\alpha \times \beta = \beta \times \alpha$, namely, it can be construed to signify

$$d \in \alpha \cap \text{cls}'\beta \text{ excl} . d' \in \beta \cap \text{cls}'\alpha \text{ excl} . \supset . \cup `d \text{ sim } \cup `d'.$$

Also incidentally the demonstration shows how to construct a cls² excl contained in $d^{\times}$; for $k \in \text{cls}^2 \text{ excl}$ and $k \supset d^{\times}$.

★8 $((a)) \in \text{cls}^3 . \supset . \therefore$

·1 $$(\cup `(a)) = q \ni \{\exists ((a)) \cap p \ni (q = \cup `p)\}. \quad \text{Df.}$$

Note: In $((a))$ the symbol a is devoid of independent signification.

·11 $\cup `(\cup `(a)) = \cup `\cup `((a))$.

·12 $p, q \in ((a)) . p \frown = q . \supset_{p,q} . \cup `p \frown = \cup `q : \supset . \mu(\cup `(a)) = \mu((a))$.

$$[\text{Hp} . x \in (\cup `(a)) . \supset . ((a)) \cap y \ni (x = \cup `y) \in 1, \quad (1)$$

$$\text{Hp} . y \in ((a)) . \supset . (\cup `(a)) \cap x \ni (x = \cup `y) \in 1, \quad (2)$$

$$(1) . (2) . \supset . \text{Prop}].$$

·2 $\text{cls}^3 \text{ excl} = \text{cls}^3 \cap \text{cls}^2 \text{ excl}$. Df.

·21 $\text{cls}^3 \text{ excl excl} = \text{cls}^3 \text{ excl} \cap \text{cls}' (\text{cls}^2 \text{ excl})$. Df.

·22 $((a)) \in \text{cls}^3 \text{ arithm} . = . (\cup `(a)) \in \text{cls}^2 \text{ excl} . ((a)) \supset \text{cls}^2 \text{ excl}$. Df.

Note: "Arithm" is contracted from "arithmetical"; this definition is essential to the statement and proof of the associative laws in their full generality.

$\star 8{\cdot}23$ $((a)) \in \mathrm{cls}^3 \mathrm{arithm} . \supset . \cup `((a)) \in \mathrm{cls}^2 \mathrm{excl}$,

$[\star 8{\cdot}22 . \mathrm{Hp} . \supset : p \in ((a)) . q, q' \in p . q \frown = q' . \supset . q \cap q' = \Lambda,$ (1)

$\star 8{\cdot}22 . \mathrm{Hp} . p, p' \in ((a)) . p \frown = p' . \supset . \cup `p \cap \cup `p' = \Lambda . \supset :$

$q \in p . q' \in p' . \supset_{q, q'} . q \cap q' = \Lambda,$ (2)

$(1) . (2) . \supset . \mathrm{Prop}]$.

$\cdot 3$ $((a)) \in \mathrm{cls}^3 \mathrm{arithm} . \supset . \sum_{p \in ((a))} \sum_{q \in p} \mu q = \mu \cup `(\cup `(a))$. Df.

$\cdot 31$ $((a)) \in \mathrm{cls}^3 \mathrm{arithm} . \supset . \sum_{p \in ((a))} \sum_{q \in p} \mu q = \sum_{q \in \cup `((a))} \mu q$, $[\star 8{\cdot}11 . \supset . \mathrm{Prop}]$.

Note: This is the associative law for addition when the number of brackets and the number of numbers in each bracket is not necessarily finite or denumerable.

$\star 9$ $((a)) \in \mathrm{cls}^3 . ((a)) \supset \mathrm{cls}^2 \mathrm{excl} . \supset . \therefore$

$\cdot 0$ $((a)^{\times}) = m \ni \{\exists ((a)) \cap p \ni (m = p^{\times})\}$. Df.

$\cdot 1$ $((a)^{\times}) = ((c)) . \supset . (\cup `(c)) = (\cup `(a))$, $[\star 6{\cdot}21 . \supset \mathrm{Prop}]$.

$\star 10$ $((a)) \in \mathrm{cls}^3 \mathrm{arithm} . \supset . \therefore$

$\cdot 0$ $\mu((a)^{\times}) = \mu((a)) = \mu(\cup `(a))$,

$[\mathrm{Hp} . \star 8{\cdot}12 . \supset . \mu((a)) = \mu(\cup `(a))$, (1)

$\mathrm{Hp} . p, q \in ((a)^{\times}) . p \frown = q . \supset . \cup `p \frown = \cup `q,$ (2)

$(2) . \star 8{\cdot}12 . \star 9{\cdot}1 . ((a)^{\times}) = ((c)) . \supset . \mu((a)^{\times}) = \mu(\cup `(c)) = \mu(\cup `(a));$ (3)

$(1) . (3) . \supset . \mathrm{Prop}]$.

$\cdot 1$ $((a)^{\times}) \in \mathrm{cls}^3 \mathrm{excl}$.

$\cdot 11$ $p, q \in ((a)^{\times})^{\times} . p \frown = q . \supset . \cup `p \frown = \cup `q,$

$[m, m' ((a)^{\times}) . \supset . \cup `m \cap \cup `m' = \Lambda,$ (1)

$\mathrm{Hp} . \supset . \exists ((a)^{\times}) \cap m \ni (p \cap m \frown = q \cap m),$ (2)

$\mathrm{Hp} . m \in ((a)^{\times}) . p \cap m \frown = q \cap m . u = \imath p \cap m .$

$v = \imath q \cap m . \supset : \exists u \frown v . \cup . \exists v \frown u,$ (3)

$(1) . \mathrm{Hp}(1) . \mathrm{Hp}(3) . \supset . u \cap \imath(q \cap m') = \Lambda . v \cap \imath(p \cap m') = \Lambda,$ (4)

$(2) . (3) . (4) . \supset : \exists \cup `p \frown \cup `q . \cup . \exists \cup `q \frown \cup `p : \supset . \mathrm{Prop}]$.

$\cdot 12$ $((a)^{\times})^{\times} = ((b)) . \supset . \mu(\cup `(b)) = \mu((a)^{\times})^{\times}$,

$[\star 8{\cdot}12 . \star 10{\cdot}11 . \supset . \mathrm{Prop}]$.

$\star 10{\cdot}13$ $\cup\,`\cup\,`((a)^{\times}) = \cup\,`\cup\,`((a))$,

$[y \in \cup\,`((a)^{\times}) . = . \mathcal{I}((a)) \cap p \ni (y \in p^{\times})$, (1)

$z \in \cup\,`\cup\,`((a)^{\times}) . = . \mathcal{I}((a)) \cap p \ni (z \in \cup\,`p^{\times})$, (2)

$(2) . \star 6{\cdot}21 . \supset : z \in \cup\,`\cup\,`((a)^{\times}) . = . \mathcal{I}((a)) \cap p \ni (z \in \cup\,`p) : \supset . \text{Prop}]$

$\cdot 131$ $q, q' \in \cup\,`((a)^{\times}) . \supset : q - = q' . \supset . \cup\,`q - = \cup\,`q'$,

$[p_1, p_2 \in ((a)) . p_1 - = p_2 . q \in p_1^{\times} . q' \in p_2^{\times} . \supset . \cup\,`q - = \cup\,`q'$ (1)

$p \in ((a)) . q, q' \in p^{\times} . p \in \text{cls}^2 \text{ excl} . \supset . \cup\,`q - = \cup\,`q'$, (2)

$(1) . (2) . \supset . \text{Prop}]$.

$\cdot 14$ $((a)^{\times})^{\times} = ((b)) . \supset . (\cup\,`(b)) = \{\cup\,`((a))\}^{\times}$,

$[\star 6{\cdot}0 . \star 10{\cdot}11 . \supset : x \in (\cup\,`(b)) . = . ((b)) \cap y \ni (x = \cup\,`y) \in 1 .$

$= . y \ni \{y \supset \cup\,`((a)^{\times}) : p \in ((a)^{\times}) . \supset_p . y \cap p \in 1 : x = \cup\,`y\} \in 1$. (1)

$\star 6{\cdot}0 . \star 10{\cdot}13 . \star 10{\cdot}131 . \supset : x \in \{\cup\,`((a))\}^{\times} . = .$

$x \supset \cup\,`\cup\,`((a)) : q \in \cup\,`((a)) . \supset_q . x \cap q \in 1 : = :$

$x \supset \cup\,`\cup\,`((a)^{\times}) : q \in \cup\,`((a)) . \supset_q . x \cap q \in 1 : = :$

$y \ni \{y \supset \cup\,`((a)^{\times}) : p \in ((a)^{\times}) . \supset_p . y \cap p \in 1 : x = \cup\,`y\} \in 1$, (2)

$(1) . (2) . \supset . \text{Prop}]$.

$\cdot 2$ $\mu((a)^{\times})^{\times} = \mu\{\cup\,`((a))\}^{\times}$, $[\star 10{\cdot}12 . \star 10{\cdot}14 . \supset . \text{Prop}]$.

$\cdot 21$ $\prod\limits_{p \in ((a))}^{q \in p} \mu q = \mu((a)^{\times})^{\times}$. Df.

$\cdot 22$ $\prod\limits_{p \in ((a))}^{q \in p} \mu q = \prod\limits_{q \in \cup\,`((a))} \mu q$, $[\star 10{\cdot}2 . = . \text{Prop}]$.

Note: This proposition, or its equivalent $\star 10{\cdot}2$, is the associative law of multiplication when the number of brackets and the number of factors in each bracket is not necessarily finite or denumerable.

$\star 11$

$\cdot 0$ $d \in \text{cls}^2 \text{ excl} . f \in \text{cls} . f \cap \cup\,`d = \Lambda . h = \iota(\cup\,`d) \cup \iota f .$

$((g)) = (\iota f \cup \iota d)^{\times} . \supset . \cup\,`((g)^{\times}) = h^{\prime}$,

$[\cup\,`((g)^{\times}) = m \ni \{f \underline{\subset} m \in 1 . d \cap p \ni (p \cap m \in 1) \in 1 . m \supset f \cup (\cup\,`d)\}$

$= m \ni \{f \cap m \in 1 . \cup\,`d \cap m \in 1 . m \supset f \cup (\cup\,`d)\} = h^{\times}]$.

Note: This is the distributive law for addition and multiplication; it can be symbolized thus:

$\cdot 1$ $\alpha \in Nc . d \in \text{cls}^2 \text{ excl} . \supset . \alpha \times \mu(\cup\,`d) = \sum\limits^{p \in d} \alpha \times \mu p$.

SECTION V.

On Powers, Combinations and Permutations.

$\star 12$ $\quad \alpha, \beta \in Nc \,.\, \supset \therefore$

$\cdot 0$ $\quad \alpha, b \in \alpha \cap \text{cls}'\beta \text{ excl} \,.\, \supset \,.\, \mu a^{\times} = \mu b^{\times},$ [$\star 7{\cdot}2 \,.\, \supset \,.$ Prop].

$\cdot 1$ $\quad \beta^{\alpha} = \imath Nc \cap x \ni \{a \in \alpha \cap \text{cls}'\beta \text{ excl} \,.\, \supset \,.\, x = \mu a^{\times}\}.$ Df.

$\star 13$ $\quad ((a)) \in \text{cls}^3 \text{ arithm} \,.\, w \in Nc : q \in^2 ((a)) \,.\, \supset \,.\, q \in w : \supset ::$

$\cdot 0$ $\quad \mu\,((a)^{\times})^{\times} = \prod_{p \in ((a))} w^{\mu p},$ [$\star 12{\cdot}1 \,.\, \supset \,.$ Prop].

$\cdot 1$ $\quad \gamma \in Nc \,.\, \cup\, `((a)) \supset \gamma \,.\, \supset \,.\, \prod_{p \in ((a))} \gamma^{\mu p} = \gamma^{\sum_{p \in ((a))} \mu p},$ [$\star 10{\cdot}22 \,.\, \supset \,.$ Prop].

$\cdot 2$ $\quad \alpha, \beta, \gamma \in Nc \,.\, \supset \,.\, (\gamma^{\beta})^{\alpha} = \gamma^{\alpha \times \beta},$

[Hp ($\star 13{\cdot}1$) $.\, ((a)) \in \alpha \cap \text{cls}'\beta \,.\, \star 13{\cdot}1 \,.\, \star 7{\cdot}3 \,.\, \supset \,.$ Prop].

$\star 14$ $\quad u, v \in \text{cls} \,.\, u \cap v = \Lambda \,.\, \supset :$

$\cdot 0$ $\quad v^u = Nc \to 1 \cap R \ni (\rho = u \,.\, \breve{\rho} \supset v).$ Df.

$\cdot 1$ $\quad \mu\,(v^u) = \mu v^{\mu u},$

$[x \in u \,.\, \supset_x \,.\, w_x = (\iota x \cup \iota v)^{\times} : w = k \ni \{\exists u \cap x \ni (k = w_x)\} : \supset \,.\, \therefore$

$p \in w^{\times} \,.\, = : x \in u \,.\, \supset_x \,.\, w_x \cap p \in 1 : p \supset \cup\, `w \,.\, \therefore \supset \,.\, \mu w^{\times} = \mu\,(v^u)$ (1)

$w \in \mu u \cap \text{cls}'\mu v \,.\, \supset \,.\, \mu w^{\times} = \mu v^{\mu u}, \ldots .$ (2) (1) $.$ (2) $.\, \supset \,.$ Prop].

Note: This proposition deduces Cantor's definition of a power [cf. Math. Ann., Bd. XLVI] from the definition of a power given in $\star 12{\cdot}1$.

$\star 15$ $\quad \alpha \in \text{cls} \,.\, \supset \therefore$

$\cdot 0$ $\quad \mu\ \text{cls}'\alpha = 2^{\mu\alpha},$

$[x \in \alpha \,.\, \supset_x \,.\, a_x = \text{cls}'\alpha \cap u \ni (x \in u) \,.\, a'_x = \text{cls}'\alpha - a_x :$

$c = k \ni \{\exists \alpha \cap x \in (k = \iota a_x \cup \iota a'_x)\} : \supset \,.\, \mu c^{\times} = 2^{\mu\alpha},$ (1)

$m \in c^{\times} \,.\, \supset \,.\, d_m = \imath \text{cls}'\alpha \cap u \ni \{x \in \alpha \,.\, \supset \,.\, \therefore a_x \in m \,.\, \supset \,.\, x \in u : a'_x \in m \,.\, \supset \,.\, x - \in u\} :$

$\supset \,.\, \text{cls}'\alpha = w \ni \{\exists c^{\times} \cap m \ni (w = d_m)\} \,.\, \supset \,.\, \mu\ \text{cls}'\alpha = \mu c^{\times},$ (2)

(1) $.$ (2) $.\, \supset \,.$ Prop].

$\cdot 1$ $\quad \beta \in Nc \,.\, \supset \,.\, C^{\alpha}_{\beta} = \text{cls}'\alpha \cap \beta.$ Df.

$\cdot 2$ $\quad \text{cls}'\alpha = x \ni \{\exists Nc \cap \beta \ni (x \in C^{\alpha}_{\beta})\}.$

$\cdot 3$ $\quad \sum_{\beta \leq \mu\alpha} \mu C^{\alpha}_{\beta} = \mu\ \text{cls}'\alpha = 2^{\mu\alpha}.$

$\cdot 4$ $\quad \alpha' \text{ sim } \alpha \,.\, \beta \in Nc \,.\, \supset \,.\, \mu C^{\alpha}_{\beta} = \mu C^{\alpha'}_{\beta}.$

$\star 16$ $\quad \alpha, \beta, \gamma \in Nc \,.\, \supset ::$

$\star 16{\cdot}0 \quad C^{\alpha}_{\beta} = \imath Nc \cap x \ni \{\alpha \in \alpha \,.\, \supset \,.\, C^{\alpha}_{\beta} \in x\}.$ Df.

$\cdot 1 \quad \sum^{\rho+\sigma=\gamma} C^{\alpha}_{\rho} \times C^{\beta}_{\sigma} = C^{\alpha+\beta}_{\gamma},$

$[\alpha \in \alpha \,.\, \beta \in \beta \,.\, \alpha \cap \beta = \Lambda \,.\, u = \gamma \cap z \ni \{\exists\, (x, y) \ni (x \supset \alpha \,.\, y \supset \beta \,.\, z = x \cup y)\} \,.$
$\supset \,.\, u = \gamma \cap z \ni (z \supset \alpha \cup \beta) \,.\, \supset \,.\, \text{Prop}].$

$\cdot 2 \quad \alpha \in Nc \text{ fin} \,.\, \supset \,.\, C^{\alpha}_{\alpha} = 1.$

$\cdot 21 \quad \alpha \in Nc \text{ infin} \,.\, \rho \in Nc \,.\, \rho \leqq \alpha \,.\, \alpha \in \alpha \,.\, \supset \,.\, C^{\alpha}_{\alpha_\rho} = C^{\alpha}_{\alpha} \cap p \in (\alpha - p \in \rho).$ Df.

Note: The class of infinite numbers denoted in this memoir by $N\alpha_0$ is defined below (cf. ★30·0); it is the class of numbers such that if β be any one of its members, $\beta + \beta = \beta$, and hence if γ be any smaller number $\beta + \gamma = \beta$. In Section III these propositions are deduced by the use of the hypothetical primitive proposition ★4·3; and in the sets of propositions ★30, ★31 and ★32, properties of the class are discussed. It will be noticed that in the proof of ★19·7 there is a forward reference to ★31·3.

$\cdot 22 \quad \alpha \in N\alpha_0 \,.\, \rho \in Nc \,.\, \rho < \alpha \,.\, \alpha \in \alpha \,.\, \supset \,.\, \mu C^{\alpha}_{\alpha_\rho} = C^{\alpha}_{\rho},$
$[\star 4{\cdot}38 \,.\, \supset \,.\, \alpha - \rho = \alpha \,.\, \supset \,.\, \text{Prop}].$

$\cdot 23 \quad \alpha \in N\alpha_0 \,.\, \alpha \in \alpha \,.\, \supset \,.\, \mu C^{\alpha}_{\alpha_\alpha} = C^{\alpha}_{\alpha}.$
$[\alpha + \alpha =: \alpha \,.\, \supset \,.\, \exists \,\text{cls}'\alpha \cap \alpha \cap u \ni (\alpha - u \in \alpha) \,.\, \supset \,.\, C^{u}_{\alpha} \supset C^{\alpha}_{\alpha_\alpha} \,.\, \supset \,.\, C^{\alpha}_{\alpha} \leqq \mu C^{\alpha}_{\alpha_\alpha}, \quad (1)$
$C^{\alpha}_{\alpha_\alpha} \supset C^{\alpha}_{\alpha} \,.\, \supset \,.\, \mu C^{\alpha}_{\alpha_\alpha} \leqq C^{\alpha}_{\alpha}, \ldots . \quad (2) \qquad (1) \,.\, (2) \,.\, \supset \,.\, \text{Prop}].$

$\cdot 24 \quad \alpha \in N\alpha_0 \,.\, \supset \,.\, C^{\lambda}_{\alpha} = 2^{\alpha}, \qquad [\alpha \in \alpha \,.\, \supset \,.\, C^{\alpha}_{\alpha} = \sum^{\rho \leqq \alpha} \mu C^{\alpha}_{\alpha_\rho} = \sum^{\rho \leqq \alpha} C^{\alpha}_{\rho} = 2^{\alpha}].$

$\cdot 3 \quad \alpha, \beta \in Nc \,.\, \supset \,.\, \alpha^{\beta} \leqq C^{\alpha\beta}_{\beta},$
$[\alpha \in \beta \cap \text{cls}'\alpha \text{ excl} \,.\, \supset \,.\, \mu \cup `a = \alpha\beta \,.\, \mu a^{\times} = \alpha^{\beta}, \quad (1)$
$a^{\times} \supset \beta \,.\, \supset \,.\, a^{\times} \supset C^{\smile`\alpha}_{\beta}, \ldots . \quad (2) \qquad (1) \,.\, (2) \supset \text{Prop}].$

$\cdot 31 \quad \alpha \times 2^{\beta} \leqq C^{\alpha\beta}_{\beta} [\alpha \in \alpha \cap \text{cls}'\beta \text{ excl} \,.\, \supset \,.\, \mu \cup `a = \alpha\beta, \quad (1)$
$b = q \ni \{\exists a \cap p \ni (q = C^{p}_{\beta})\} \,.\, \supset \,.\, b \in \alpha \cap \text{cls}' 2^{\beta} \,.\, \supset \,.\, \mu \cup `b = \alpha \times 2^{\beta}, \quad (2)$
$\cup `b \supset C^{\smile`\alpha}_{\beta}, \ldots . \quad (3) \qquad (1) \,.\, (2) \,.\, (3) \,.\, \supset \,.\, \text{Prop}].$

Binomial Theorem.—The next three sets of props (under integral numbers 17 to 19) are devoted to proving forms of the Binomial Theorem. The main results are given in ★17·32 and ★17·4 and ★18·53, ★19·4 and ★19.7.

$\star 17 \quad \alpha, \beta, \delta, \nu \in Nc \,.\, R \in 1 \to 1 \,.\, \rho \in \alpha \cap \text{cls}'\beta \text{ excl} \,.\, \breve{\rho} \in \alpha \cap \text{cls}'\delta \text{ excl} \,.$
$\cup `\rho \cap \cup `\breve{\rho} = \Lambda \,.\, a = p \ni \{\exists \rho \cap q \ni (p = q \cup \imath\breve{\rho}q)\} \,.\, \supset \,.\, \therefore$

$\cdot 0 \quad \alpha \in \alpha \cap \text{cls}'(\beta + \delta).$

$\cdot 1 \quad a^{\times} \in (\beta + \delta)^{\alpha} \cap \text{cls}'\alpha.$

$\star 17{\cdot}11 \quad m \in a^{\times} . q \in \rho . \supset . m \cap q \in 1 \cup 0.$

$\cdot 2 \quad a_{\rho_\nu} = a^{\times} \cap m \ni \{ \cup {}^{`}\rho \cap m \in \nu \}$ Df.

$\cdot 21 \quad l_\nu \in C^\rho_\nu . \supset . \breve{l}_\nu = \breve{\rho} - \breve{\rho}(l_\nu).$ Df.

$\cdot 211 \quad l_\nu \in C^\rho_\nu . \supset . \mu \breve{l}_\nu \in \sigma \ni (\sigma + \nu = \alpha).$

$\cdot 212 \quad l_\nu \in C^\rho_\nu . \supset . \mu \breve{l}^{\times}_\nu = \delta^{\mu \breve{l}_\nu}$

$\cdot 22 \quad a_{\rho_\nu} = m \ni [\exists (p, q) \ni \{ \exists C^\rho_\nu \cap \iota_\nu \ni (p \in \iota_\nu^{\frown} . q \in l^{\times}_\nu . m = p \cup q) \}].$

$\cdot 23 \quad \mu a_{\rho_\nu} = \sum^{l_\nu \in C^\rho_\nu} \mu l^{\times}_\nu \times \mu l^{\times}_\nu = \sum^{l_\nu \in C^\rho_\nu} \beta^\nu \times \mu l^{\times}_\nu = \beta^\nu_{\times} \sum^{l_\nu \in C^\rho_\nu} \delta^{\mu \breve{l}_\nu}.$

$\cdot 30 \quad a_\rho = k \ni \{ \exists Nc \cap \nu \ni (k = a_{\rho_\nu}) \}.$ Df.

$\cdot 31 \quad a_\rho \in \mathrm{cls}^2 \text{ excl } . \cup {}^{`}a_\rho = a^{\times}.$

$\cdot 32 \quad (\beta \dotplus \delta)^\alpha = \mu a^{\times} = \mu \cup {}^{`}a_\rho = \sum^{\nu \leqq \alpha} \mu a_{\rho_\nu} = \sum^{\nu \leqq \alpha} \beta^\nu \times \sum^{l_\nu \in C^\rho_\nu} \delta^{\mu \breve{l}_\nu}$

$\cdot 4 \quad \delta = 1 . \supset . (1+\beta)^\alpha = \sum^{\nu \leqq \alpha} \beta^\nu \times \mu C^\rho_\nu = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \beta^\nu.$

$\star 18 \quad \mathrm{Hp}(\star 17) . \alpha \in Nc \text{ fin } . \supset :$

$\cdot 1 \quad \nu \leqq \alpha . \supset . \sigma \ni (\sigma + \nu = \alpha) \in 1 . u \breve{l}_\nu = \alpha - \nu.$

[The second part of the prop follows from $\star 17{\cdot}211$].

$\cdot 2 \quad \sum^{l_\nu \in C^\rho_\nu} \delta^{\mu \breve{l}_\nu} = C^\alpha_\nu \times \delta^{\alpha - \nu}.$

$\cdot 3 \quad (\beta + \delta)^\alpha = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha - \nu}.$

$\star 19 \quad \mathrm{Hp}(\star 17) . \alpha \in N\alpha_0 . \supset . \therefore$

$\cdot 0 \quad \nu < \alpha . \sigma + \nu = \alpha . \supset . \sigma = \alpha.$

$\cdot 1 \quad \nu < \alpha . \supset . \sum^{l_\nu \in C^\rho_\nu} \delta^{\mu \breve{l}_\nu} = C^\alpha_\nu \times \delta^\alpha,$

$[\star 17{\cdot}211 . \star 19{\cdot}0 . \supset . \mathrm{Prop}].$

$\cdot 2 \quad \nu \leqq \alpha . \supset : l_\alpha \in C^\rho_{\alpha_\nu} . = . \mu \breve{l}_\alpha = \nu.$

$\cdot 3 \quad \sum^{l_\alpha \in C^\rho_\alpha} \delta^{\mu \breve{l}_\alpha} = \sum^{\nu \leqq \alpha} \mu C^\rho_{\alpha_\nu} \times \delta^\nu = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \delta^\nu = (1+\delta)^\alpha.$ $[\star 17{\cdot}4 . \supset . \mathrm{Prop}]$

$\cdot 4 \quad (\beta + \delta)^\alpha = \sum^{\nu < \alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha - \nu} + \beta^\alpha (1+\delta)^\alpha,$

$[\star 17{\cdot}32 . \star 19{\cdot}3 . \supset . \mathrm{Prop}].$

$\star 19{\cdot}7$ $(\beta+\delta)^\alpha = \delta^\alpha \times (1+\beta)^\alpha + \beta^\alpha \times (1+\delta)^\alpha$,

$[(\beta+\delta)^\alpha = \sum^{\nu<\alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha-\nu} + \beta^\alpha \times (1+\delta)^\alpha$

$= \sum^{\nu<\alpha} C^\alpha_\nu \times \delta^\nu \times \beta^{\alpha-\nu} + \delta^\alpha \times (1+\beta)^\alpha$, (1)

$\sum^{\nu=\alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha-\nu} = \delta^\alpha \sum^{\nu<\alpha} C^\alpha_\nu \times \beta^\nu \leqq \delta^\alpha \sum^{\nu\leqq\alpha} C^\alpha_\nu \times \beta^\nu \leqq \delta^\alpha \times (1+\beta)^\alpha$. (2)

$\sum^{\nu<\alpha} C^\alpha_\nu \times \delta^\nu \times \beta^{\alpha-\nu} \leqq \beta^\alpha (1+\delta)^\alpha$, (3)

$\star 31{\cdot}3 \,.\supset : \beta^\alpha \times (1+\delta)^\alpha + \delta^\alpha \times (1+\beta)^\alpha = \beta^\alpha \times (1+\delta)^\alpha \,.\cup\,.$

$\beta^\alpha \times (1+\delta)^\alpha + \delta^\alpha \times (1+\beta)^\alpha = \delta^\alpha \times (1+\beta)^\alpha$, (4)

$(1)\,.\,(2)\,.\,(3)\,.\,(4)\,.\supset.$ Prop].

$\star 20{\cdot}1$ $\alpha \in N\alpha_0 \,.\, \beta, \delta \in Nc \,.\, \beta \geqq \delta \,.\supset.\, (\beta \times \delta)^\alpha = \beta^\alpha$,

$[\alpha \in N\alpha_0 \,.\, \beta \geqq \delta \,.\supset.\, \beta^\alpha = \beta^{\alpha+\alpha} = (\beta\times\beta)^\alpha \geqq (\beta\times\delta)^\alpha$, (1)

$(\beta\times\delta)^\alpha \geqq \beta^\alpha$, (2)

$(1)\,.\,(2)\,.\supset.$ Prop].

$\star 21$ $\alpha \in N\alpha_0 \,.\, \beta \in Nc \,.\supset.\,\therefore$

$\cdot 1$ $(2\beta)^\alpha = \beta^\alpha (1+\beta)^\alpha$, [$\star 19{\cdot}7 \,.\, \beta = \delta \,.\supset.$ Prop].

$\cdot 2$ $(1+\beta)^\alpha = (2\beta)^\alpha + (1+\beta)^\alpha$, [$\star 19{\cdot}7 \,.\, \delta = 1 \,.\supset.$ Prop].

$\cdot 3$ $(2\beta)^\alpha = (1+\beta)^\alpha$, [$\star 21{\cdot}1 \,.\, \star 21{\cdot}2 \,.\supset.$ Prop].

$\cdot 4$ $\nu \in Nc$ fin $\,.\supset.\, \nu^\alpha = 2^\alpha$,

$[\sigma \in Nc$ fin $- \iota 0 \,.\, \beta = 2^\sigma + 1 \,.\, \star 21{\cdot}3 \,.$ Induct $\,.\supset.$

$(2^{\sigma+1}+2)^\alpha = (2^\sigma+2)^\alpha = (2+2)^\alpha = 3^\alpha$, (1)

$(2+2)^\alpha = 2^{2\alpha} = 2^\alpha$, (2)

$\nu \in Nc$ fin $- \iota 0 \,.\, 2^{\sigma+1} + 2 \geqq \nu \geqq 2^\sigma + 2 \,.\supset.\, (2^{\sigma+1}+2)^\alpha \geqq \nu^\alpha \geqq (2^\sigma+2)^\alpha$, (3)

$(1)\,.\,(2)\,.\,(3)\,.\supset.$ Prop].

$\star 22$ $\alpha, \beta \in Nc \,.\, \alpha \geqq \beta \,.\, a \in \alpha \,.\, b \in \beta \,.\supset.\,\therefore$

$\cdot 0$ $P^\alpha_b = 1 \to 1 \cap R \ni (\rho = b \,.\, \breve{\rho}\,`\,a)$. Df.

$\cdot 01$ $P^\alpha_\beta = \imath Nc \cap \gamma \ni (a \in \alpha \,.\, b \in \beta \,.\supset.\, P^\alpha_b \ni \gamma)$. Df.

$\cdot 02$ $P^\alpha_b \,`\, a^b$, [$\star 14{\cdot}0 \,.\, \star 22{\cdot}0 \,.\supset.$ Prop].

$\cdot 03$ $P^\alpha_\beta \leqq \alpha^\beta$, [$\star 14{\cdot}1 \,.\, \star 22{\cdot}02 \,.\supset.$ Prop].

$\cdot 04$ ${}^\beta P_\beta = \imath Nc \cap \gamma \ni \{1 \to 1 \cap R \ni (\rho = b \,.\, \breve{\rho} = b) \in \gamma\}$. Df.

$\cdot 05$ ${}^\beta P_\beta \leqq P^\beta_\beta$.

$\cdot 06$ $\beta \in Nc$ fin $\,.\supset.\, {}^\beta P_\beta = P^\beta_\beta$.

$\star 22{\cdot}1 \quad {}^{\beta}P_{\beta} \times C^{\alpha}_{\beta} = P^{\alpha}_{\beta},$

$[\alpha \in \alpha . b \in \beta : l \in C^{\alpha}_{\beta} . \supset_{l} . k_l = 1 \to 1 \cap R \ni (\rho = b . \breve{\rho} = l) :$

$k = p \ni \{\exists C^{\alpha}_{\beta} \cap l \ni (p = k_l)\} : \supset . \mu k = \mu C^{\alpha}_{\beta} = C^{\alpha}_{\beta} .$

$k \supset {}^{\beta}P_{\beta} . \cup `k = \beta^{\alpha}_{b} : \supset P_{\beta} = \mu P^{\alpha}_{b} = \mu \cup `k = {}^{\beta}P_{\beta} \times C^{\alpha}_{\beta}].$

$\cdot 2 \quad \alpha \in N\alpha_0 . \beta \in Nc \text{ fin} . \cup . \beta \in N\alpha_0 . \alpha \geqq \beta : \supset . C^{\alpha \times \beta}_{\beta} = P^{\alpha \times \beta}_{\beta} = \alpha^{\beta},$

$[\star 16{\cdot}3 . \star 22 . 03 . \star 22{\cdot}1 . \supset . \alpha^{\beta} \leqq C^{\alpha \times \beta}_{\beta} \leqq P^{\alpha \times \beta}_{\beta} \leqq (\alpha \times \beta)^{\beta}, \quad (1)$

$\text{Hp} . \star 4{\cdot}42 . \star 20{\cdot}1 . \supset . \alpha^{\beta} = (\alpha \times \beta)^{\beta}, \ldots . (2) \quad (1) . (2) . \supset . \text{Prop}],$

$\cdot 21 \quad \alpha \in N\alpha_0 . \beta \in Nc \cap \beta' \ni (\alpha \times \beta' = \alpha) . \supset . C^{\alpha}_{\beta} = P^{\alpha}_{\beta} = \alpha^{\beta},$

$[\star 22{\cdot}2 . \supset . \text{Prop}].$

$\cdot 22 \quad \alpha \in N\alpha_0 . \alpha^2 = \alpha . \supset . \alpha^{\alpha} = 2^{\alpha}, \quad [C^{\alpha}_{\alpha} = 2^{\alpha} . \star 22{\cdot}21 . \supset . \text{Prop}].$

$\star 23{\cdot}1 \quad \beta \in Nc . \supset . \beta < 2^{\beta},$

$[\alpha \in \beta \cap \text{cls}' 2 \text{ excl} . q, q' \in a^{\times} . q \cap q' = \Lambda : p \in a . \supset_{p} .$

$p' = \imath a^{\times} \cap m \ni (m = q - p \cup q' \cap p) : b = x \ni \{\exists a \cap p \ni (x = p')\} : \supset .$

$b \supset a^{\times} . \mu b = \mu a : \supset . m a^{\times} \geqq \mu a, \quad (1)$

$\alpha \in \beta \cap \text{cls}' 2 \text{ excl} . S \in 1 \to 1 . \sigma = \alpha . \breve{\sigma} \supset a^{\times} : p \in a . \supset . p_1 = \imath (p \cap \iota \breve{\sigma} p) .$

$p_2 = \imath (p - \iota p_1) : q = x \ni \{\exists a \cap p \ni (x 1' p_2)\} : \supset . \therefore$

$q \in a^{\times} : p \in a . \supset_{p} . p_2 \in q - \breve{\sigma} p . \therefore \supset . q \in a^{\times} - \breve{\sigma} a, \quad (2)$

$(1) . (2) . \supset . \text{Prop}].$

Note: This proposition is the extension for any cardinal number of Cantor's proof [cf. Jahresber. d. Deutsch. Math. Ver. I, S. 77] that the power of the continuum (2^{α_0}) is greater than the first infinite cardinal (α_0). The adaptation of the proof and its reduction to symbols are due to Russell.

In Section IV and the present Section V, all the proofs involving any characteristic properties of the class *Nc* infin have been limited to the class $N\alpha_0$ and based on the theorem that

$$\alpha \in N\alpha_0 . \beta \leqq \alpha . \supset . \alpha + \beta = \alpha.$$

It becomes important therefore to investigate for what class contained within *Nc*, the theorem

$$\beta \leqq \alpha . \supset . \alpha + \beta = \alpha$$

can be asserted, since the universal validity of $\star 4{\cdot}3$ is not proved. Furthermore, we have not succeeded in deducing, even from $\star 4{\cdot}3$, the allied proposition

$$\alpha \in Nc \text{ infin} . \beta \leqq \alpha . \supset . \alpha \times \beta = \alpha.$$

The nearest approach to it as yet is $\star 20{\cdot}1$.

The sets of propositions with integral numbers 30 to 32 will be devoted to these investigations; the proofs of some propositions practically proved by Russell in Section III will be omitted. In these sets of propositions (30 to 32) it will be found that the only properties of an *Nc* infin used are that [cf. ★4·1]

$$\alpha \in Nc \text{ infin} . \supset . \alpha \geq \alpha_0 . \alpha + \alpha_0 = \alpha ,$$

where α_0 is the first cardinal number.

★30

·0 $N\alpha_0 = Nc \cap \beta \ni [\exists Nc \cap \delta \ni (\delta > 0 . \beta = \alpha_0 \times \delta)]$. Df.

Note: As long as all the numbers considered belong to $N\alpha_0$, the propositions in Section III, ★4·31 to 4·51, hold without modification.

·1 $\alpha \in N\alpha_0 . \beta \in Nc . \supset . \therefore \alpha \geq \beta . \supset . \alpha + \beta = \alpha : \alpha \leq \beta . \supset . \alpha + \beta = \beta ,$

$[\alpha \geq \beta . \supset . \alpha \leq \alpha + \beta \leq \alpha + \alpha \leq \alpha . \supset . \alpha + \beta = \alpha ,$ (1)

$\alpha \leq \beta . \supset . \exists Nc \cap \gamma \ni (\alpha + \gamma = \beta) . \supset . \alpha + \beta = \alpha + \alpha + \gamma = \alpha + \gamma = \beta ,$ (2)

$(1) . (2) . \supset .$ Prop].

·2 $\beta , \gamma \in Nc . \exists N\alpha_0 \cap \alpha \ni [\beta \leq \alpha \leq \gamma] . \supset . \beta + \gamma = \gamma ,$

$[\gamma \leq \beta + \gamma \leq \alpha + \gamma \leq \gamma . \supset .$ Prop].

Note: This theorem is important, and enables the enunciations of some theorems of ★21 and ★22 to be extended.

·3 $\beta \in Nc . \alpha \in N\alpha_0 \cap z \ni (z \geq \beta) . \supset . \alpha \geq \alpha_0 \times \beta ,$

$[\alpha \geq \beta . \supset . \alpha_0 \times \alpha \geq \alpha_0 \times \beta . \supset . \alpha \geq \alpha_0 \times \beta]$.

·4 $\beta \in Nc . \gamma \in Nc \cap z \ni (\beta \leq \gamma \leq \alpha_0 \times \beta) . \supset . \alpha_0 \times \beta = \alpha_0 \times \gamma ,$

[★30·3 $. \supset .$ Prop].

·5 $\alpha , \alpha' \in N\alpha_0 . \alpha < \alpha' . \beta \in Nc \cap \beta \ni (\alpha = \alpha_0 \times \beta) .$

$\beta' \in Nc \cap \beta' \ni (\alpha' = \alpha_0 \times \beta') . \supset . \beta \leq \alpha_0 \times \beta < \beta' \leq \alpha_0 \times \beta' ,$

[★30·4 $. \supset .$ Prop].

★31

·1 $Nc \exp = Nc \cap \alpha \ni [\exists Nc \text{ infin} \cap \beta \ni \{\exists Nc \cap \gamma \ni (\gamma > 1 . \alpha = \gamma^\beta)\}]$. Df.

·11 $u \in \text{cls infin} . \supset . \mu \text{ cls}' u \in Nc \exp ,$

[★15·0 $. \supset .$ Prop].

·2 $\alpha \in Nc \exp . \supset . \alpha > \alpha_0 .$

·3 $Nc \exp . \supset . N\alpha_0 ,$

$[\gamma^\beta \in Nc \exp . \supset . \gamma > 1 . \supset . \gamma^{\alpha_0} \in Nc \exp . \supset . \gamma^{\alpha_0} > \alpha_0 ,$ (1)

$(1) . \supset . \gamma^\beta \leq \alpha_0 \times \gamma^\beta \leq \gamma^{\alpha_0} \times \gamma^\beta \leq \gamma^{\alpha_0 + \beta} ,$ (2)

$\gamma^\beta \in Nc \exp . \supset . \beta \in Nc \text{ infin} . \supset . \alpha_0 + \beta = \beta ,$ (3)

$(2) . (3) . \supset . \gamma^\beta = \alpha_0 \times \gamma^\beta . \supset .$ Prop].

★32

·1 $Nc\ \exp^{\alpha_0} = N\alpha_0 \cap \alpha \ni [\exists N\alpha_0 \cap \beta \ni \{\exists Nc \cap \gamma \ni (\gamma > 1 . \alpha = \gamma^\beta)\}]$, Df.

·2 $\beta \in Nc\ \exp^{\alpha_0} . \supset . \beta^2 = \beta$,
$[\beta = \gamma^\delta . \supset . \beta^2 = \gamma^{2\delta} = \gamma^\delta = \beta]$.

·3 $\beta \in Nc . \gamma \in Nc\ \exp^{\alpha_0} . \beta \leq \gamma . \supset . \beta\gamma = \gamma$.

·4 $\alpha \in Nc\ \exp^{\alpha_0} . \beta \in Nc - \iota 0 - \iota 1 . \beta \leq 2^\alpha . \supset . \beta^\alpha = 2^\alpha$,
$[2^\alpha \leq \beta^\alpha \ldots . (1)$, Hp . ★32·2 . $\supset . \beta^\alpha \leq (2^\alpha)^\alpha \leq 2^{\alpha^2} \leq 2^\alpha$, (2)
(1) . (2) . $\supset$. Prop$]$.

·5 $\delta \in Nc\ \exp^{\alpha_0} . \gamma \in Nc - \iota 0 - \iota 1 . \alpha \in Nc\ \exp^{\alpha_0} . \delta \geq \alpha . \beta = \gamma^\delta . \supset . \beta^\alpha = \beta$,
$[\beta^\alpha = \gamma^{\alpha \times \delta} = \gamma^\delta = \beta]$.

On a Method of Constructing all the Groups of Order p^m.

By G. A. Miller.

A group G of order p^m (p being any prime) contains at least one invariant subgroup H of order p^{m-1}. If G is represented as a regular substitution group, then H has p systems of intransitivity and its p transitive constituents $(H_1, H_2, \ldots, H_p)$ are transformed cyclically by every substitution of G which is not contained in H. Moreover, there is a substitution t of order p which is commutative with every substitution of H and permutes its transitive constituents cyclically.* It seems desirable, in considering the construction of all the possible groups of order p^m, to divide them into two classes: 1). Those in which the operators of H are transformed only according to the group (I_0) of cogredient isomorphisms of H, and 2). Those in which these operators are transformed according to a larger group.

If G belongs to the first class, it must involve a substitution s which is commutative with each substitution of H and is not contained in H. It may be supposed that t permutes the transitive constituents of H in the same manner as s does. The substitution $st^{-1} = s'$ transforms each of these constituents into itself, and is commutative with every substitution of H. That is, s' is one of the substitutions of the group of order $p^{(m-1)_p}$ which is composed of all the substitutions which are commutative with each substitution of H without permuting any of its transitive constituents or involving any letters not found in H.

The group generated by H and $s't$ can be transformed into one generated by H and s_1t, s_1 being a substitution of H_1 which is commutative with every substitution of H_1. Furthermore, the $p-1$ groups of order p^m generated by H and the following substitutions, taken separately,

$$s_1t, \quad s_1^2t, \quad \ldots, \quad s_1^{p-1}t,$$

* Cf. Quarterly Journal of Mathematics, vol. XXVIII, 1896, p. 237.

are conjugate, since all of them may be obtained from any one of them by permuting systems of intransitivity of H, the first system being left unchanged. On transforming the group generated by H and t by $r_1^{p-1} r_3 r_4^2 \ldots r_p^{p-2}$, r_1 being any invariant substitution of H_1 while $r_a (a = 2, 3, \ldots, p)$ is the transform of r_1 with respect to t^a, there results a group generated by H and $r_1^{-p} t$.

From what precedes, it may be observed that all the possible groups of order p^m, which include H and transform its operators only according to substitutions of I_0, may be obtained by multiplying H by $s_1 t$, where s_1 must satisfy all of the following conditions:* 1). It must be an invariant substitution of H_1; 2). If we form the quotient group of these invariant substitutions with respect to the subgroup generated by their p^{th} powers, then it is not necessary to use for s_1 more than one from each of the divisions corresponding to the operators of this quotient group; 3). Not more than one from each system of conjugate operators of H_1 under its holomorph should be used for s_1; 4). The $p-1$ groups generated by H and one of the following operators $s_1 t, s_1^2 t, s_1^3 t, \ldots, s_1^{p-1} t$ are conjugate; that is, if one substitution is a power of another, it is not necessary to use both for s_1 unless one of them is identity.

When G contains an operator which transforms H according to a substitution (i) which is not contained in I_0, then G transforms the operators of H according to a group I_1 whose order is p times the order of I_0 and which includes I_0. Let $t_1 t_2 \ldots t_p$ transform the constituents of H according to i. Since i^p must be in I_0, $(t_1 t_2 \ldots t_p)^p$ must transform H in the same manner as one of its own substitutions s' transforms H. Suppose that s' is commutative with $t_1 t_2 \ldots t_p$ and that $s_1, s_2, \ldots, s_p$ represent its constituents contained in $H_1, H_2, \ldots H_p$ respectively. The substitution $t_1^{-p} s_1 = s_1'$ is commutative with every substitution of H. Hence $s_1' t_1 t_2 \ldots t_p t$ transforms the substitutions of H in exactly the same manner as $t_1 t_2 \ldots t_p$ does. Moreover, $(s_1' t_1 t_2 \ldots t_p t)^p = s'$, From this it follows that H and $s_1' t_1 t_2 \ldots t_p t$ generate a group in which H is transfomed in the required manner.

The assumption that there is a substitution s' in H which is commutative with $t_1 t_2 \ldots t_p$ and transforms H in exactly the same way as $(t_1 t_2 \ldots t_p)^p$ does, is clearly a consequence of the existence of G and hence does not limit the generality of the method. If H is fully known, its group of isomorphisms I can

* Cf. Transactions of the American Mathematical Society. vol. II, 1901, p. 264. It should be observed that H is not necessarily Abelian in the present article.

readily be determined, and the subgroups which satisfy the conditions imposed upon I_1 can be found. In fact, it is only necessary to determine one of the subgroups whose order is the highest power of p that divides the order of I, since only one from each system of such subgroups as I_1, which are conjugate under I, needs to be considered. Furthermore, it is necessary to consider only the substitutions which may correspond to any particular one of the operators (not in I_0) of such a subgroup. It remains, therefore, only to examine all the possible groups which may be obtained by replacing $s_1' t_1 t_2 \ldots . t_p t$ by other substitutions which transform it in the same manner.

Any one of these substitutions may be transformed into $s_1' t_1 t_2 \ldots . t_p t$ multiplied by a substitution which involves only the letters of H_1 and is commutative with each of the substitutions of H_1. In order that the p^{th} power of such a product may be in H_1, it is necessary that this multiplier is an invariant substitution of H_1 and is also commutative with t_1. Let any one of these products be denoted by $s_1'' t_1 t_2 \ldots . t_p t$. No new group is obtained by using in place of s_1'' any of its conjugates under the subgroup of the group of isomorphisms of H_1 which includes all its operators that are commutative with t_1. When H_1 is Abelian, these conjugates include all the powers of s_1'' which are prime to its order.*

Let r_1 be an invariant substitution of H_1, and let

$$t_1 r_1 t_1^{-1} = u_1 r_1, \quad t_1 u_1 t_1^{-1} = u_1, \quad u_1^p = 1, \quad p > 2,$$

where u_1 does not necessarily differ from identity. Hence

$$(r_1^{p-1} r_3 r_4^2 \ldots . r_p^{p-2})^{-1} s_1' t_1 t_2 \ldots . t_p t r_1^{p-1} r_3 r_4^2 \ldots . r_p^{p-2}$$
$$= s_1' r_1^{1-p} r_2 r_3 \ldots . r_p u_2 u_3^2 u_4^3 \ldots . u_p^{p-1} t_1 t_2 \ldots . t_p t.$$

Transforming succsseively by u_p^{p-1}, u_{p-1}^{2p-3}, ... and multiplying into $(r_1 r_2 \ldots r_p)^{-1}$ there results $s_1' r_1^{-p} t_1 t_2 \ldots . t_p t$. Hence no additional group is obtained by multiplying $s_1' t_1 t_2 \ldots . t_p t$ by the p^{th} power of such an r_1. That is, if the quotient group of the invariant substitutions of H_1, which are commutative with t_1, are arranged with respect to the p^{th} powers of such substitutions as r_1, then no more than one substitution from each of these divisions needs to be used as a multiplier of $s_1' t_1 t_2 \ldots . t_p t$.

When $p = 2$, the results of the preceding paragraph remain true, provided u_1 is either 1 or a power of r_1. In the latter case the order of r_1 must exceed 4.

* Transactions of the American Mathematical Society, vol. II, 1901, p. 260.

When $p > 3$, these results still remain true if u_1 is replaced by any substitution (y_1) such that $t_1 y_1 t_1^{-1} = x_1 y_1$, $t_1 x_1 t_1^{-1} = x_1$, $x_1^p = y_1^p = 1$. It should be observed that it is not proved that all the groups of order p^m obtained, in the above manner, from a particular group of order p^{m-1} are distinct. In general, additional limitations must be imposed upon s''. However, the special considerations for small values of m are sufficiently limited to make the method practicable. For instance, when $m < 5$ and $p > 3$, only one non-Abelian group will occur twice if we use for the subgroup of order p^{m-1} the different Abelian groups of order p^{m-1}.

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Non-Euclidean Properties of Plane Cubics and of their First and Second Polars.

By Henry Freeman Stecker.

This paper is in continuation of a paper* on the same subject and assumes a knowledge of the terms and results given there.

We derive first one or two theorems which come simply from results of the former paper. Consider relation (B), p. 35, of the paper mentioned. This relates to focal lines, but to such pairs as do not pass through the same absolute point. For any focal line there are six others which have no absolute point in common with it. Consider relation (B) for two of these in connection with the original focal lines. Then that ratio becomes

$$\frac{\sin\frac{\overline{xa_2}}{2ki}\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xa_4}}{2ki}\sin\frac{\overline{xc}}{2ki}} = \text{const.} \qquad (1)$$

Hence: *If we consider two focal lines through the same absolute points, and the chords joining their third intersections to the third intersection of any focal line having no absolute point in common with either of them, then if the ratio of the sines of the N. E. D. from any point of a cubic to these two chords be multiplied by the ratio of the sines of the N. E. D. from the same point of the cubic to the two focal lines, this product is constant for all points of the cubic.*

Again, if we consider (B) in relation to a pair of directors. Then, from (33) and (B), we find

$$\frac{\sin^2 m}{\sin^2 n}\cdot\frac{\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xc}}{2ki}} = \text{const.}, \qquad (2)$$

* American Journal of Mathematics, vol. XXII.

where m and n are the N. E. D. from any point of the cubic to one director of either pair. Hence: *The ratio of the square of the sines of the N. E. D. from any point on a cubic to two directors (one in each pair) multiplied by the ratio of the sines of the N. E. D. from any point of a cubic to the poles of the chords of third intersections, this product is constant for all points of the cubic.*

Consider any two points, O and P, on any secant line cutting a plane cubic in points c_i. From any pair of directors, d_1 and d_2, whose focal lines have no common absolute point, let fall perpendiculars, p_1 and p_2, upon this secant line. Let a_i and b_i be the lines joining c_i with d_1 and d_2 respectively. Connect O with d_1 and d_2 by lines β, γ; also P with the same points by lines α, δ. Then

$$\left(\sin \frac{a_i}{2ki}\right)^2 = \left(\sin \frac{x_i}{2ki}\right)^2 + \left(\sin \frac{p_1}{2ki}\right)^2,$$

where p_i is the intercept between c_i and the foot of p_1. Likewise,

$$\left(\sin \frac{b_i}{2ki}\right)^2 = \left(\sin \frac{y_i}{2ki}\right)^2 + \left(\sin \frac{p_2}{2ki}\right)^2,$$

where y_i is the intercept from c_i to the foot of p_2.

But the ratio of the left-hand member of these equations is a constant λ for all points of the plane cubics. This gives:

$$\sin^2 \frac{x_i}{2ki} + \sin^2 \frac{p_1}{2ki} = \lambda \sin^2 \frac{y_i}{2ki} + \lambda \sin^2 \frac{p_2}{2ki},$$

and from these equations we find:

$$\begin{vmatrix} \sin^2 \frac{x_1}{2ki} & \sin^2 \frac{y_1}{2ki} & 1 \\ \sin^2 \frac{x_2}{2ki} & \sin^2 \frac{y_2}{2ki} & 1 \\ \sin^2 \frac{x_3}{2ki} & \sin^2 \frac{y_3}{2ki} & 1 \end{vmatrix} = 0. \tag{3}$$

Or, changing the form a little, we may say:

The expression $\sum_{i \neq j}^{1,2,3} \left[\prod \sin^2 \frac{x_i}{2ki} \sin^2 \frac{y_j}{2ki} \right]$ *vanishes for all points of a plane cubic.*

If we put $OP = \sin^{-1} a$ and $OP_i = \sin^{-1} a_i$, also write $(aa_m) \equiv \begin{vmatrix} a & \sqrt{1-a^2} \\ a_m & \sqrt{1-a_m^2} \end{vmatrix}$, we find, when P is on the second polar of O with respect to the cubic

$$\sum_{i \neq j \neq k}^{1,2,3} \sin^{-1} a_i \sin^{-1} a_j \sin^{-1} (aa_k) = 0, \tag{4}$$

and if P is on the first polar,

$$\sum_{i \neq j \neq k}^{1,2,3} \sin^{-1} a_i \sin^{-1} (aa_j) \sin^{-1} (aa_k) = 0. \tag{5}$$

We find also:

$$\sin oP = \begin{vmatrix} (\beta i) & (\beta 1) \\ (\alpha i) & (\alpha 1) \end{vmatrix}, \tag{6}$$

and

$$\sin \overline{oc_i} = \begin{vmatrix} \cos \frac{x_i}{2ki} & (\beta 1) \\ \sin \frac{x_i}{2ki} & (\beta i) \end{vmatrix},$$

where $(hk) \equiv \sqrt{\sin^2 \frac{h}{2ki} + k^2 \sin^2 \frac{p_1}{2ki}}$.

Combining these, we are able to say:

When P is on the second polar of O, the expression

$$\sum_{i \neq j \neq k}^{1,2,3} \left\{ \sin^{-1} \begin{vmatrix} \cos \frac{x_i}{2ki} & (\beta 1) \\ \sin \frac{x_i}{2ki} & (\beta 1) \end{vmatrix} \sin^{-1} \begin{vmatrix} \cos \frac{x_j}{2ki} & (\beta 1) \\ \sin \frac{x_j}{2ki} & (\beta 1) \end{vmatrix} \right.$$
$$\left. \sin^{-1} \left(\begin{vmatrix} (\beta i) & (\beta 1) \\ (\alpha i) & (\alpha 1) \end{vmatrix} \begin{vmatrix} \cos \frac{x_k}{2ki} & (\beta 1) \\ \sin \frac{x_k}{2ki} & (\beta i) \end{vmatrix} \right) \right\}$$

vanishes for all points of the cubic. If P is on the first polar, the above expression, after the second factor is replaced by $\sin^{-1} \left(\begin{vmatrix} (\beta i) & (\beta 1) \\ (\alpha i) & (\alpha 1) \end{vmatrix} \begin{vmatrix} \cos \frac{x_k}{2ki} & (\beta 1) \\ \sin \frac{x_k}{2ki} & (\beta i) \end{vmatrix} \right)$ *vanishes for all points of the cubic.*

Consider the tangent to the first polar through P. Call θ_1 the angle which α makes with oP; θ_{2i} the angle which α makes with a_i. Let d' and d'' be the two directors of the first polar and r_1 and r_2 the distances, respectively, from d' and d'' upon the tangent through P. Also r_1 and r_2 meet the tangent in points n_1 and n^2. Take θ_3 the angle made by the tangent and $\overline{n_1 d_1}$; θ_4 by $\overline{n_1 d_1}$ and $\overline{P d_1}$; θ_5 by the tangent and $\overline{Pd'}$; θ_6 by r_1 and $\overline{Pd'}$. Let ϑ_i be the corresponding angles for d_2 and d''. Then if we write H_θ for

$$\frac{\sin\frac{\theta_1}{2k'}\sin\frac{\theta_3}{2k'}\sin\frac{\theta_5}{2k'}}{\sin\frac{\theta_{2i}}{2k'}\sin\frac{\theta_4}{2k'}\sin\frac{\theta_6}{2k'}}$$

we find:

$$\sin\frac{a_i}{2ki} = H_\theta \sin\frac{r_1}{2ki},$$

$$\sin\frac{b_i}{2ki} = H_\vartheta \sin\frac{r_2}{2ki},$$

and from these:

$$\frac{H_\theta}{H_\vartheta}\,\frac{\sin\frac{r_1}{2ki}}{\sin\frac{r_2}{2ki}} = \lambda, \qquad (\lambda = \text{constant}).$$

But $\sin\frac{r_1}{2ki}\sin\frac{r_2}{2ki}$ is also constant, so that we may state: *The expression* $\dfrac{H_\theta \sin\frac{r_1}{2ki}}{H_\vartheta \sin\frac{r_2}{2ki}}$ *is constant for all points of a cubic.* a_i meets OP and divides it into two parts, say m_i and n_i; the latter on the side of P. Then we find from the above: (8). $\sin m_i = \sigma \sin \pi_{1i}\sqrt{H_\theta H_\vartheta}$; σ is constant and π_{1i} the angle made by a_i and β.

We have also:

$$\cos\frac{a_i}{2ki} = \cos\frac{m_i}{2ki}\lambda_1 + \lambda_2 \sin\frac{a_i}{2ki}\sin\frac{m_i}{2ki},$$

$$\cos\frac{b_i}{2ki} = \cos\frac{m_i}{2ki}\lambda_3 + \lambda_4 \sin\frac{b_i}{2ki}\sin\frac{m_i}{2ki},$$

where λ_i are constants.

From these, remembering that the ratio of the sines or cosines of a_i and b_i' is constant, we find:

$$\tan \frac{m_i}{2ki} = \tau \csc \frac{a_i}{2ki}, \qquad (\tau = \text{constant}).$$

Similarly:

$$\cos \frac{b_i}{2ki} = \cos \frac{n_i}{2ki} \cos \frac{\delta}{2ki} + \cos \frac{\vartheta_1}{2k'} \sin \frac{b_i}{2ki} \sin \frac{n_i}{2ki},$$

$$\cos \frac{a_i}{2ki} = \cos \frac{n_i}{2ki} \cos \frac{\alpha}{2ki} + \cos \frac{\theta_1}{2k'} \sin \frac{a_i}{2ki} \sin \frac{n_i}{2ki}.$$

From these we find:

$$\sin \frac{n_i}{2ki} = \operatorname{ctn} \frac{a_i}{2ki} \; \frac{\lambda \cos \frac{\delta}{2ki} - \cos \frac{\alpha}{2ki}}{\lambda \cos \frac{\delta}{2ki} \cos \frac{\theta_1}{2k'} - \cos \frac{\alpha}{2ki} \cos \frac{\vartheta_1}{2k'}},$$

$$\cos \frac{n_i}{2ki} = \cos \frac{a_i}{2ki} \; \frac{1 - \cos \frac{\vartheta_1}{2ki}}{\lambda \cos \frac{\delta}{2ki} \cos \frac{\theta_1}{2k'} - \cos \frac{\alpha}{2ki} \cos \frac{\vartheta_1}{2k'}},$$

where λ is constant.

From these we find $\cos a_i$: First write:

$$A \equiv 1 - \cos \frac{\vartheta_1}{2k'}; \quad B = \lambda \cos \frac{\delta}{2ki} - \cos \frac{\alpha}{2ki},$$

and

$$C = \lambda \cos \frac{\delta}{2ki} \cos \frac{\theta_1}{2k'} - \cos \frac{\alpha}{2ki} \cos \frac{\vartheta_1}{2k'}.$$

Then: $$\cos a_i = \frac{1}{C} \sqrt{\tfrac{1}{2}(A^2 + B^2 + C^2 + \sqrt{(A^2 + B^2 + C^2)^2 - 4A^2C^2}},$$

or say: $$\cos a_i = \frac{D_1}{C},$$

and $\sin a_i = \dfrac{D_2}{C}$, where $D_2 = \sqrt{\tfrac{1}{2}(C^2 - A^2 - B^2 - \sqrt{(A^2 + B^2 + C^2) - 4A^2C^2})}$.

Then we have:

$$\frac{m_i}{2ki} = \tan^{-1} \tau \frac{C}{D_2}, \text{ say } \tan^{-1} M_{1i},$$

$$\frac{n_i}{2ki} = \cos^{-1} \frac{AD_1}{C^2}, \text{ say } \cos^{-1} M_{2i}.$$

Then, for all points of a first polar, we have:

$$\sum_{i \neq j}^{1,2,3} \frac{\tan^{-1} M_{1i}}{\cos^{-1} M_{2i}} \frac{\tan^{-1} M_{1j}}{\cos^{-1} M_{2j}} = 0,$$

and for all points of a second polar:

$$\sum_{i \neq j}^{1,2,3} \frac{\tan^{-1} M_{1i}}{\cos^{-1} M_{2i}} = 0.$$

Consider next P as on the first polar of O with regard to the cubic; also d and d'' a pair of directors of this first polar. Let P be joined to each by lines c_1 and c_2 respectively. Let c_1 make with $\overline{d'd_1}$ an angle ξ_1; also $\overline{od'}$ an angle ξ_1' with oP. Let the corresponding angles for d'' be ξ_2 and ξ_2'. Then:

$$\sin \frac{c_i}{2ki} = \frac{\sin \frac{\xi_i'}{2k'}}{\sin \frac{\xi_i}{2k'}} \sin \frac{OP}{2ki}, \qquad (i = 1, 2).$$

But $\frac{c_1}{2ki} \pm \frac{c_2}{2ki}$ is constant, say equal to A, so that

$$\sin^{-1}\left(\frac{\sin \frac{\xi_i'}{2ki}}{\sin \frac{\xi_1}{2k'}} \sin \frac{OP}{2ki}\right) \pm \sin^{-1}\left(\frac{\sin \frac{\xi_2'}{2k'}}{\sin \frac{\xi_2}{2k'}} \sin \frac{OP}{2ki}\right) = A,$$

Solving this for $\sin \frac{OP}{2ki}$, we have:

The sine of the N. E. D. from pole to first polar with regard to any cubic is equal to:

$$\frac{\lambda_1 \sin \frac{\xi_1}{2k'} \sin \frac{\xi_2}{2k'}}{\left[\sin^2 \frac{\xi_1'}{2k'} \sin^2 \frac{\xi_2}{2k'} + \sin^2 \frac{\xi_2'}{2k'} \sin^2 \frac{\xi_1}{2k'} \pm \lambda_2 \sin \frac{\xi_1'}{2k'} \sin \frac{\xi_2'}{2k'}\right]^2},$$

where λ_1 and λ_2 are constants, such that $\lambda_1^2 + \lambda_2^2 = 1$; and the $\pm$ to mean that when one sign is used the other cannot be. The angle ξ_i and ξ_i' have the meaning given above.

π_{1i} was the angle which a_i makes with β and π_{2i} the angle made by b_i and γ. Then

$$\sin\frac{m_i}{2ki} = \sin\frac{\pi_{1i}}{2k'}\,\frac{\sin\dfrac{a_i}{2ki}}{\sin\dfrac{P_1}{2k'}} = \sin\frac{\pi_{2i}}{2k'}\,\frac{\sin\dfrac{b_i}{2ki}}{\sin\dfrac{P_2}{2k'}},$$

where P_1 and P_2 are angles made by OP with β and γ respectively. Then since $\dfrac{\sin\dfrac{a_i}{2ki}}{\sin\dfrac{b_i}{2ki}}$ is constant for all points of a cubic, we have: *If O is a fixed point, then the ratio of the sines of the angles formed by lines from O to a pair of directors, with lines joining these directors to a point of the cubic is constant for all points of the cubic.*

Let the ratio $\dfrac{\sin\dfrac{\pi_{1i}}{2k'}}{\sin\dfrac{\pi_{2i}}{2k'}}$ be the constant K; also the ratio $\dfrac{\sin\dfrac{P_2}{2k'}}{\sin\dfrac{P_1}{2k'}} = \lambda_1$. Then we find:

$$\frac{\sin^2\dfrac{\pi_1}{2k'} - \sin^2\dfrac{P_1}{2k'}\sin^2\dfrac{m_i}{2ki}}{\lambda^2\sin^2\dfrac{\pi_1}{2k'} - \sin^2\dfrac{P_1}{2k'}\sin^2\dfrac{m_i}{2ki}} = \frac{\lambda^2}{\lambda_1^2},$$

and from this we find:

$$\sin\frac{m_i}{2ki} = \tau\sin\frac{\pi_{1i}}{2k'},$$

where τ is constant and equal to $csc\,\dfrac{P_2}{2k'}\sqrt{\dfrac{\lambda^4-\lambda_1^2}{\lambda^2-\lambda_1^2}}$. Then, for n_i, we find:

$$\sin\frac{n_i}{2k'} = \frac{\sin\dfrac{\pi_{1i}}{2k'}}{\sin\dfrac{s_1}{2k'}}\sin\frac{a_i}{2ki} = \frac{\sin\dfrac{\pi_{2i}}{2k'}}{\sin\dfrac{s_2}{2k'}}\sin\frac{b_i'}{2ki},$$

where s_1 is the angle made by n_i and α; s_2 that by n_i and δ.

And hence:

$$\sin \frac{n_i}{2ki} = csc \frac{\pi_{2i}}{2k'} csc \frac{s_1}{2k'} \sqrt{\frac{\lambda^2 \sin^4 \frac{\pi_{2i}}{2k'} - \sin^4 \frac{s_1}{2k'}}{\lambda^2 - 1}},$$

and from these we get:

The expressions:

$$\sum_{1,2,3} \frac{\sin^{-1}\tau \sin \frac{\pi_{1i}}{2k'}}{\sin^{-1} \sqrt{\frac{\lambda^2 \sin^4 \frac{\pi_{2i}}{2k'} - \sin^4 \frac{s_1}{2k'}}{(\lambda^2 - 1) \sin^2 \frac{\pi_{2i}}{2k'} \sin^2 \frac{s_1}{2k'}}}}$$

and

$$\sum_{i \neq j}^{1,2,3} \frac{\sin^{-1}\tau \sin \frac{\pi_{1i}}{2k'}}{\sin^{-1} \sqrt{\frac{\lambda^2 \sin^4 \frac{\pi_{2i}}{2k'} - \sin^4 \frac{s_1}{2k'}}{(\lambda^2 - 1) \sin^2 \frac{\pi_{2i}}{2k'} \sin^2 \frac{s_1}{2k'}}}} \cdot \frac{\sin^{-1}\tau \sin \frac{\pi_{1j}}{2k'}}{\sin^{-1} \sqrt{\frac{\lambda^2 \sin^4 \frac{\pi_{2j}}{2k'} - \sin^4 \frac{s_1}{2k'}}{(\lambda^2 - 1) \sin^2 \frac{\pi_{2j}}{2k'} \sin^2 \frac{s_1}{2k'}}}}$$

vanish for all points of the second polar, respective first polar of any cubic.

Let the tangent to the first polar at P meet a pair of focal lines of this polar in points x_1 and x_2. Represent the distance from O to x_1 and x_2 by l_1 and l_2 respectively.

Let l_i make an angle ω_i with OP. From a non-euclidean property of the conic the distance $\overline{Px_1}$ equals Px_2 or say $t_1 = t_2$. Then we have:

$$\cos \frac{t_1}{2ki} = \cos \frac{OP}{2ki} \cos \frac{l_1}{2ki} + \cos \frac{\omega_1}{2k'} \sin \frac{t_1}{2ki} \sin^2 \frac{OP}{2ki},$$

$$\cos \frac{t_2}{2ki} = \cos \frac{OP}{2ki} \cos \frac{l_2}{2ki} + \cos \frac{\omega_2}{2k'} \sin \frac{t_2}{2ki} \sin \frac{OP}{2ki},$$

and from these:

$$\cos \frac{OP}{2ki} \left(\cos \frac{l_2}{2ki} - \cos \frac{l_1}{2ki} \right) = \sin \frac{OP}{2ki} \sin \frac{t_1}{2ki} \left(\cos \frac{\omega_1}{2ki} - \cos \frac{\omega_2}{2k'} \right)$$

or

$$\tan\frac{OP}{2ki}=\frac{\cos\frac{l_2}{2ki}-\cos\frac{l_1}{2ki}}{\sin\frac{t_1}{2ki}\left(\cos\frac{\omega_1}{2k'}-\cos\frac{\omega_2}{2k'}\right)},$$

To eliminate $\sin\frac{t_1}{2ki}$, we have:

$$\cos\frac{2t_1}{2ki}=\cos\frac{l_1}{2ki}\cos\frac{l_2}{2ki}+\sin\frac{l_1}{2ki}\cos\frac{(\omega_1+\omega_2)}{2k'}\sin\frac{2t_1}{2ki},$$

and from this we find:

$$\sin\frac{t_1}{2ki}=\sqrt{\tfrac{1}{2}\left(\frac{1-M+s^2\pm s\sqrt{1-M^2+s^2}}{1+s^2}\right)},$$

where $M=\cos\frac{l_1}{2ki}\cos\frac{l_2}{2ki}$ and $s=\sin\frac{l_1}{2ki}\cos\left(\frac{\omega_1+\omega_2}{2k'}\right)$ so that $\sin\frac{t_1}{2ki}$ is a function of l_i and ω_i only. Represent this function by $\frac{1}{\Theta(l_i\omega_i)}$, then we may state:

If O is the pole of a point P on the first polar with regard to a cubic, and if the tangent to this polar at P meet the focal lines in two points, then l_1 and l_2, the distances from O to these points and the angles ω_1 and ω_2, which they make with OP, are connected by the relation:

$$\tan\frac{OP}{2ki}=\frac{\cos\frac{l_2}{2ki}-\cos\frac{l_1}{2ki}}{\cos\frac{\omega_1}{2k'}-\cos\frac{\omega_2}{2k'}}\Theta(l_i\omega_i).$$

In the same configuration, if we consider a pair of directors of the cubic, we get a simple relation between angles.

The angles are θ_i and ϑ_i $(i=1, 2, 3, 4)$, where θ_4 is made by OP and α; θ_3 by a_i and OP; while θ_1 and θ_2 are made at r_1 by t_1 and α respectively; r_1 being the line from x_1 to d_1; ϑ_i are corresponding angles for d_2 and x_2. Then we find:

$$\sin\frac{a_i}{2ki}=\frac{\sin\frac{\theta_4}{2k'}\sin\frac{\theta_1}{2k'}}{\sin\frac{\theta_3}{2k'}\sin\frac{\theta_2}{2k'}}\sin\frac{t_1}{2ki}$$

and

$$\sin\frac{b_i}{2ki} = \frac{\sin\frac{\vartheta_4}{2k'}\sin\frac{\vartheta_1}{2k'}}{\sin\frac{\vartheta_3}{2k'}\sin\frac{\vartheta_2}{2k'}}\sin\frac{t_2}{2ki}.$$

From these: *The ratio of* $\dfrac{\sin\frac{\theta_4}{2k'}\sin\frac{\theta_1}{2ki}}{\sin\frac{\theta_3}{2k'}\sin\frac{\theta_2}{2k'}}$ *to* $\dfrac{\sin\frac{\vartheta_4}{2k'}\sin\frac{\vartheta_1}{2k'}}{\sin\frac{\vartheta_3}{2k'}\sin\frac{\vartheta_2}{2k'}}$ *is constant for all points of a cubic.*

The remaining theorems of the previous paper all involve three distances or the ratios of such, and the developments from these are, it seems, complicated.

CORNELL UNIVERSITY, ITHACA, N. Y.

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